Partial Differential Equations

General Analytic Theory

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0: Structure, Notation, and Classification

Reference for this section is (pg.701, [5])

Structure of Differential Operators:

Real, n-dimensional Euclidean space, denoted $\boldsymbol{E^n}$ or $\mathbb{R}^{\boldsymbol{n}}$ (in coordinate form), has *vector space structure*, *metric-topological structure*, *inner product stucture* (hence *norm structure*), and the traditional structure of an *algebra*.

It is *complete* with respect to its metric, so the **partial operators** are well defined, provided we feed them at least locally continuous functions. Recall:

$$\partial_i f := \lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}.$$

We may **compose** partial operators, yielding **higher order derivative operators**:

$$D^{\alpha} := \partial_1^{\alpha_1} \circ \dots \circ \partial_n^{\alpha_n},$$

where $\alpha := (\alpha_1, \dots, \alpha_n)$ is an **multi-index** of **size** $|\alpha| := \sum_{i=1}^{n} \alpha_i$.

Notes: These operators require more of our function inputs of course (they must be $|\alpha|$ times differentiable); The only size 1 operators are just the individual partials; Each D^{α} is a **linear operator**, since each ∂_i is.

In contrast to this last note, we have another binary operation defined on the D^{α}'s that yields **non-linearity**. Namely, the **tensor product**. Let us define:

$$D^{\alpha} \otimes D^{\beta}(f, g) := D^{\alpha}(f) \cdot D^{\beta}(g),$$

for *appropriately differentiable* functions f, g. As a special case, we may consider the situation with $\alpha = \beta$ and $f = g$:

$$(D^{\alpha}(f))^2 := D^{\alpha} \otimes D^{\alpha}(f, f).$$

This generalizes of course to **k-fold tensor products**, $(D^{\alpha}(f))^k$.

These may also be combined via linear combinations to give **polynomials** and **power series** in each of the D^{α} atoms. Lastly, the coefficients in such linear combinations can be given **spatial dependence**, **lower order derivative operator dependence**, and/or **function argument dependence**. For example, one such term could look like:

$$\lambda\big(x, \{D^{\beta}(f) : |\beta| < |\alpha|\}, f\big) \cdot D^{\alpha}(f)$$

or equivalently:

$$\lambda\big(x, D^{|\alpha|}(f), \dots, D(f), f\big) \cdot D^{\alpha}(f),$$

provided we define the sets:

$$D^k := \{D^{\beta} : |\beta| = k\}.$$

(Next Page)

On Shorthand Notation:

• Def: In the special cases where $\boldsymbol{k} = \mathbf{1}$ and $\boldsymbol{k} = \mathbf{2}$, the sets $\boldsymbol{D}^{\mathbf{1}}$ and $\boldsymbol{D}^{\mathbf{2}}$ can be displayed nicely in a vector (resp. matrix). These displays are given aliases and special symbols. Namely: $\boldsymbol{\nabla}$ (the **gradient vector**) and $\boldsymbol{\nabla}^{\mathbf{2}}$ (the **Hessian matrix**). These two are used very frequently. So for clarity:

$$\boldsymbol{\nabla} := (\partial_1^{(1)}, ..., \partial_n^{(1)})$$

and

$$\boldsymbol{\nabla}^2 := \begin{pmatrix} \partial_{11}^{(2)} & \cdots & \partial_{1n}^{(2)} \\ \vdots & \ddots & \vdots \\ \partial_{n1}^{(2)} & \cdots & \partial_{nn}^{(2)} \end{pmatrix}.$$

Note: In general, we may summarize the sets, $\boldsymbol{D}^{\boldsymbol{k}}$, in multi-dimensional arrays.

• Def: Some other common notation involves what is called the **Laplacian operator**, which is just:

$$\boldsymbol{\Delta} := \sum_{i=1}^{n} \partial_{ii}.$$

• Def: (p.66 [5]) The ***d′Alembertian*** operator is a short hand for the wave equation operator:

$$\square := \partial_{tt}^{(2)} - \boldsymbol{\Delta}$$

[Exercise: Create your own symbol and slap your name on it!]

(Next page)

On Classification:

We've already encountered ways to distinguish differential operators. We have the **order of the operator** determined by the highest value of $\boldsymbol{k} = |\boldsymbol{\alpha}|$ for multi-index $\boldsymbol{\alpha}$ appearing within the expression. We also have **linear, multi-linear, semi-linear, quasi-linear, and nonlinear** (depending on how the atoms $\boldsymbol{D^\alpha}$ appear in the expression).

When we move to operator equations, we have a couple more things that we can say.

• Def: A **partial differential equation** is ultimately of the form:

$$\boldsymbol{F}\left(\boldsymbol{x}, \boldsymbol{u}, \{\partial_i\}_{i \in \{1,\ldots,n\}}\right) = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$$

taken over some domain, $\boldsymbol{U} \subseteq \mathbb{R}^{\boldsymbol{n}}$ of an unknown function $\boldsymbol{u} : \boldsymbol{U} \to \mathbb{R}$.

We further describe operator equations as **homogeneous** if the RHS is zero. Otherwise, we say it is **non-homogeneous**. *I don't particularly like the homogeneous terminology because of its appearance in the discussion of symmetric polynomials where the terms all have the same order. As well, we can subtract the RHS over and create a new operator... Nevertheless we will use it.*

Other classifications involve having **constant coefficients**, **spatially dependent coefficients**, etc. As well, when the operator takes on special model equation forms that resemble equations for **quadric surfaces** or **conic sections**, we give them classifications such as **parabolic, elliptic, hyperbolic** etc. Some authors save these terms for relations involving the discriminant of a change of variables etc. I would wait until you get there to worry about that.

• Def: **(Uncoupled) Systems of partial differential equations** have the form:

$$\left\{\boldsymbol{F}^j\left(\boldsymbol{x}, \boldsymbol{u}^j, \{\partial_i\}_i\right) = \boldsymbol{f}^j\left(\boldsymbol{x}, \boldsymbol{u}^j\right)\right\}_{\boldsymbol{j}}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$, vary up to the $\boldsymbol{n}$ in $\mathbb{R}^n$. This is essentially a collection of PDEs whom must all simultaneously be satisfied. The collection of functions $\boldsymbol{u^j}$ may be taken to be the same function over and over again. The systems are further classified as **coupled** if there exists an equation that depends on more than one function variable $\boldsymbol{u^j}$. Think diagonal matrix vs. non-diagonal.

Lastly, **initial** or **boundary data** may be added to the problems. These usually serve to narrow the solution sets after they have been found (for example when undetermined coefficients exist). **Cauchy**, **Dirichlet**, and **Neumann** are names that specify the type of auxiliary data for the problem. Again, better to consider these later.

Breathe!

I: Solutions Catalog (a.k.a. The Maw)

*This list was adapted from (pgs.3-6 [5]). Although each equation has **homogeneous** and **non-homogeneous** counterparts, as well as **auxiliary data** (initial or boundary conditions), the main focus is on finding solutions (not necessarily unique ones). These solutions are typically explicit, given up to undetermined coefficients.*

*Handling non-homogeneity and auxiliary data/constraints should be treated separately in theory. At least for the Linear case, **superposition** may be applied to handle both problems.*

Links to each article are given below. The template for each article is as follows:

- *Equation,*
- *Constraints,*
- *Solution (not nec. unique),*
- *Solution Technique, and a*
- *REALITY CHECK (direct solution verification).*

LINEAR EQUATIONS

1.) **Transport**
2.) **Laplace**
3.) **Heat/Diffusion**
4.) **Wave (★)**
5.) **PLANAR WAVE SOLUTIONS - Collection of PDEs**
> Satisfies: Heat, Wave, Klein-Gordon, Schrodinger, and Airy.

More: Helmholtz, Liouville, Holmogorov, Fokker-Planck, Telegraph, General Wave, and Beam Eq.

NONLINEAR EQUATIONS

6.) **Conservation Laws**
More: Eikonal, Poisson, p-Laplacian, Minimal Surface, Monge-Ampere, Inviscid Burger, Scalar Reaction-Diffusion, Porous Medium, Wave, Korteweg-deVries, and Schrodinger Eq.

LINEAR SYSTEMS

7.) **System from Stokes Theorem**
More: Equilibrium Eq.'s of Linear Elasticity, Evolution Eq.'s of Linear Elasticity, and Maxwell's Eq's.

NONLINEAR SYSTEMS

System of Convervation Laws, Reaction-Diffusion System, Euler's Eq.'s for Incrompressible Inviscid Flow, and Navier-Stokes Eq.'s for Incompressible Viscious Flow.

Breathe!

1a. (Homogeneous) Transport Equation (p.18-19 [5])

Domain and Unknown Function:

$$U_{open} \subseteq \mathbb{R}^n \times (0, \infty) \quad \text{and} \quad u : \bar{U} \to \mathbb{R}$$

Equation:

$$u_t + b \cdot Du = 0 \quad \text{for } (x, t) \in U$$

Constraints:

$$u = g \quad \text{for } (x, 0) \in \partial U$$

where $b \in \mathbb{R}^n$ is a fixed known direction, and $g : \partial U \to \mathbb{R}$ is a fixed known function on the boundary.

Solution:

$$u\big((x, t)\big) := g(x - tb).$$

Solution Technique:

Apply the theory and algorithm from **Section II.1.1**, this is done on the next page.

(Continues)

Solution Technique:

1.) We start by checking our problem is actually F.O.L. so that the M.o.C. applies.

$$u_t + b \cdot Du \;=\; \sum_{i=1}^{n} b^i \partial_i u + 1 \cdot \partial_t u - 0 \;=\; \sum_{i=1}^{n+1} \lambda_i \partial_i u - \lambda_{n+2}. \;\square$$

Here we note that our problem is *homogeneous* ($\lambda_{n+2} \equiv 0$) and we call $t \leftrightarrow x_{n+1}$.

2.) Define the (n+2)-vector fields:

$$\eta := < \nabla u, -1 > \qquad \text{and} \qquad \Lambda := < b, 1, 0 >$$

Then we note the problem factors into: $\langle \Lambda, \eta \rangle = 0$. Continue.

3-5.) *Before we move on, let us rename our usual curve parameter to s, so that we have $\gamma_p(s)$ instead to not confuse with the (n+1)th coordinate.*

Let $x \in \partial U$ be arbitrary, then $x := (x_1, ..., x_n, t = 0)$ and $p := (x, u(x))$.

We write down the system then:

$$\partial_s \gamma_p^i(s) = \Lambda^i|_p \qquad i \in \{1, ..., n+2\}$$
$$\gamma_p(0) = p$$
$$\gamma_p^{n+2}(0) = g(x)$$

which instantiates to:

$$\partial_s \gamma_p^i(s) = b^i \qquad \text{for } i \in \{1, ..., n\}$$
$$\partial_s \gamma_p^{n+1}(s) = 1$$
$$\partial_s \gamma_p^{n+2}(s) = 0$$
$$\gamma_p(0) = (x_1, ..., x_n, 0, u(x))$$
$$\gamma_p^{n+2}(0) = g(x) \qquad \text{(boundary condition)}$$

Integrating the first three lines w.r.t. the parameter, s, we get:

$$\gamma_p^i(s) = b^i \cdot s + C^i \qquad \text{for } i \in \{1, ..., n\}$$
$$\gamma_p^{n+1}(s) = s + C^{n+1}$$
$$\gamma_p^{n+2}(s) = C^{n+2}$$

for introduced constants $C^i \in \mathbb{R}$.

Solution Technique (Continued):

Applying the initial and boundary conditions, we solve for the new constants:

$$C^i = x_i, \qquad \text{for } i \in \{1, ..., n\}$$

$$C^{n+1} = 0$$

$$C^{n+2} = g(x)$$

- -

Combining these results, the integral curve through the arbitrary boundary point is:

$$\gamma_p(s) = \big(b^1 s + x_1 \ , \ ... \ , \ b^n s + x_n \ , \ s \ , \ g(x)\big)$$

6a.) Projecting the curve to $\mathbb{R}^{n+1}$, we have:

$$\widetilde{\gamma}_p(s) = (b^i s + x_i, s).$$

Any $x' \in Im(\widetilde{\gamma}_p(s))$, corresponds to an $s' \in \mathbb{R}$. Opening this statement up gives:

$$(x'_1, ..., x'_{n+1}) = \big(\widetilde{\gamma}^1_p(s'), ..., \widetilde{\gamma}^{n+1}_p(s')\big).$$

Which in our case is just:

$$(x'_1, ..., x'_{n+1}) = \big(b^1 s' + x_1, ..., b^n s' + x_n, s'\big).$$

$$\implies \ (x_1, ..., x_{n+1}) = \big(- b^1 s' + x'_1, ..., -b^n s' + x'_n, 0\big).$$

Note $t' =: x'_{n+1} = s'$ says nothing about x_{n+1} which we know is zero by virtue of $x \in \partial U$. But it will be useful shortly.

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6b.) Now if we take a step back, according to the theory, solutions along the projected curves look like:

$$u(x') = \gamma_p^{n+2}(s') = g(x).$$

What we did in part (a) was eliminate the expression for the boundary point:

$$u(x') = g\Big(\big(- b^1 s' + x'_1, ..., -b^n s' + x'_n, 0\big)\Big).$$

Which is otherwise written: $u((x', t')) = g((x' - bt'), 0) =: g(x' - bt')$.■

REALITY CHECK:

Let's make sure the solution obtained actually satisfies the PDE!

$$\boldsymbol{u(x,t) = g(x - bt)}$$

vs.

$$\boldsymbol{u_t + b \cdot Du = 0}$$

We have by the chain rule:

$$\boldsymbol{\partial_t u = \partial_t\big(g(x - bt)\big) = \sum_{i=1}^{n} \partial_i g(x - bt) \cdot (-b^i)}$$

and

$$\boldsymbol{\partial_j u = \partial_j\big(g(x - bt)\big) = \sum_{i=1}^{n} \partial_i g(x - bt) \cdot \delta^i_j = \partial_j g(x - bt)}.$$

Hence:

$$\boldsymbol{b \cdot Du = Du \cdot b = \sum_{i=1}^{n} \partial_i g(x - bt) \cdot (+b^i)}$$

and the result is blatant. ■

1b. (Non-Homogeneous) Transport Equation (p.19 [5])

Domain and Unknown Function:

$$U_{open} \subseteq \mathbb{R}^n \times (0, \infty) \qquad \text{and} \qquad u : \bar{U} \to \mathbb{R}$$

Equation:

$$u_t + b \cdot Du = f \qquad \text{for } (x, t) \in U$$

Constraints:

$$u = g \qquad \text{for } (x, 0) \in \partial U$$

where $f : U \to \mathbb{R}$ is a fixed known function, $b \in \mathbb{R}^n$ is a fixed known direction, and $g : \partial U \to \mathbb{R}$ is a fixed known function.

Solution:

$$u\big((x,t)\big) := g(x - tb) + \int_0^t f(x + (s-t)b, s)ds.$$

Solution Technique:

Apply the algorithm/theory from **Section II.1**. Next page.

(Continues)

Solution Technique:

1.) We know step (1) is satisfied from the previous problem (1a).
2.) Call $t =: x_{n+1}$. Define the (n+2)-vector fields which are evaluated at points in $x' \in U$ but located at $p' = (x', u(x'))$ on the graph, Γu:

$$\eta(x')\big|_{p'} :=< \nabla\big(u(x')\big), -1 > \big|_{p'} \qquad \text{and} \qquad \Lambda(x')\big|_{p'} :=< b, 1, f(x') > \big|_{p'}$$

Then we note the problem factors into: $\langle \Lambda, \eta \rangle = 0$. Continue.

3-5.) Take a boundary point $p = (x, u(x)) \in \partial(\Gamma u)$. Then the integral curve, $\gamma_p(s) : S_x \subseteq \mathbb{R} \hookrightarrow \Gamma u$, through p is given by the solution to:

$$\dot{\gamma}_p^i(s') = \Lambda^i\big(x' = \widetilde{\gamma}_p(s')\big) \qquad \text{for} \qquad i \in \{1, ..., n+2\}$$

$$\gamma_p(0) = p \qquad \text{(init.)}$$
$$\gamma_p^{n+2}(0) = g(x) \qquad \text{(bdry.)}$$

- -

Substitute and integrate:

$$\gamma_p^i(s') = b^i * s' + C^i \qquad \text{for} \qquad i \in \{1, ..., n\}$$

$$\gamma_p^{n+1}(s') = s' + C^{n+1}$$

$$\gamma_p^{n+2}(s') = \int f\big(\widetilde{\gamma}(s')\big) ds' + C^{n+2}$$

Applying the initial and boundary conditions gives:

$$x_i = \gamma_p^i(0) = b^i * (0) + C^i \qquad \text{for} \qquad i \in \{1, ..., n\}$$

$$0 = \gamma_p^{n+1}(0) = (0) + C^{n+1}$$

$$g(x) = \gamma_p^{n+2}(0) = \Big[\int f\big(\widetilde{\gamma}(s')\big) ds'\Big]\Big|_{s'=0} + C^{n+2}$$

And this says our integral curve is:

$$\gamma_p(s') = \left(b^i * s' + x_i \ , \ s' \ , \ \int f\big(\widetilde{\gamma}(s')\big) ds' + g(x) - \Big[\int f\big(\widetilde{\gamma}(s')\big) ds'\Big]\Big|_{s'=0} \right)$$

which can be cleaned up by combining the integrals into:

$$\gamma_p(s') = \left(b^i * s' + x_i \ , \ s' \ , \ g(x) + \int_{w=0}^{w=s'} f\big(\widetilde{\gamma}(w)\big) dw \right)$$

Solution Technique (Continued):

6.) The theory says then that our solution is of the form:

$$u(x') = \gamma_p^{n+2}(s') = g(x) + \int_{w=0}^{w=s'} f\big(\widetilde{\gamma}(w)\big)dw$$

So we just need to eliminate the boundary point x and s' in favor of $x' = (x', t') \in U$. Accordingly,

$$x' = (x'_1, ..., x'_{n+1}) = \widetilde{\gamma}_p(s') = (b^i * s' + x_i, s')$$

gives us:

$$x_i = (x'_i - b^i s', 0) \qquad \text{and} \qquad t' =: x'_{n+1} = s'.$$

Applying this yields:

$$u(x') = g((x'_i - b^i t', 0)) + \int_0^{t'} f\big((b^i w + [x'_i - b^i t'], w)\big)dw$$

which when rearranging is:

$$u(x') = g((x'_i - b^i t', 0)) + \int_0^{t'} f\big((x'_i + (w - t')b^i, w)\big)dw$$

Now we've rid reference to x *as a boundary point so define* $x' =: (x, t)$, *drop the primes, and call* $w \leftrightarrow s$ *(and identify bdry points with last component zero):*

$$u((x,t)) = g((x_i - b^i t, 0)) + \int_0^t f\big((x_i + (s-t)b^i, s)\big)ds$$

$$\implies u(x,t) = g(x - bt) + \int_0^t f\big(x + (s-t)b, s\big)ds \;. \blacksquare$$

REALITY CHECK:

Let's make sure the solution obtained actually satisfies the PDE!

$$u(x,t) = g(x - bt) + \int_0^t f(x + (s-t)b, s)ds$$

vs.

$$u_t + b \cdot Du = f$$

By the second Results Entry, we have:

$$\partial_t \int_0^t f(s,t)ds = f(t,t) + \int_0^t \partial_t f(s,t)ds,$$

where we take:

$$f(s,t) := f(x + (s-t)b, s).$$

So that expanding with the chain rule in the second term, we get:

$$\partial_t \int_0^t f(x + (s-t)b, s)ds = f(x + (t-t)b, t) + \int_0^t \Big[\sum_{i=1}^n \partial_i f(x + (s-t)b, s) \cdot \partial_t (x^i + (s-t)b^i)\Big] ds$$

$$= f(x,t) - \int_0^t \big[b \cdot Df(x + (s-t)b, s)\big] ds \qquad (\star)$$

- -

Spatial derivatives pass through the integral as time is constant to them, so we don't have to work so hard. Coupling with linearity of the integral, we arrive at:

$$b \cdot D\Big[\int_0^t f(x + (s-t)b, s)ds\Big] = \int_0^t \big[b \cdot Df(x + (s-t)b, s)\big] ds. \qquad (\star\star)$$

- -

Noting that $g(x - bt)$ satisfies the homogeneous equation, the desired result follows:

$$u_t + b \cdot Du = (\star) + (\star\star) = f(x,t). \qquad \blacksquare$$

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2. (Homogeneous) Laplace Equation (p.20-25 [5])

Domain and Unknown Function:

$$U_{open} \subseteq \mathbb{R}^n \qquad \text{and} \qquad u : \bar{U} \to \mathbb{R}$$

Equation:

$$\Delta u := \sum_{i=1}^{n} \partial_{ii} u = 0 \qquad \text{for } x \in U$$

Constraints:

$$u = g \qquad \text{for } x \in \partial U$$

where $g : \partial U \to \mathbb{R}$ is a fixed known function.

Solution:

$$u(x) := \begin{cases} C_1 r + C_2 & \text{if } n = 1 \\ C_1 ln|r| + C_2 & \text{if } n = 2 \\ C_1 r^{2-n} + C_2 & \text{if } n \geq 3 \end{cases}$$

for constants C_i and parameter $r := |x|_2$. With $r \neq 0$. The constraint has not been applied.

Solution Technique:

Apply theory from Section II.2.1-3.

(Continues)

Solution Technique:

Suppose (using the 2-norm) that the solution is a function of a radial parameter:

$$r := |x|_2 := \sqrt{(x_1)^2 + \ldots + (x_n)^2}.$$

That is: $u = u(r)$ satisfies $\Delta u = 0$.

- -

Let us compute the Laplacian. Further assume $r \neq 0$. Using the Chain Rule and the *Power Rule* yields:

$$\partial_i u(r) = \partial_r u \cdot \partial_i r = u_r \cdot \frac{1}{2\sqrt{(x_1)^2 + \ldots + (x_n)^2}} \cdot 2x_i = u_r \cdot \frac{x_i}{r}.$$

Taking the second derivative then requires the *Product Rule*:

$$\partial_{ii} u(r) = \partial_i \left(u_r \cdot \frac{x_i}{r} \right) = \partial_i(u_r) \cdot \frac{x_i}{r} + u_r \cdot \partial_i(x_i) \cdot \frac{1}{r} + u_r \cdot x_i \cdot \partial_i \left(\frac{1}{r} \right)$$

$$= u_{rr} \cdot (\partial_i r) \cdot \frac{x_i}{r} + u_r \cdot (1) \cdot \frac{1}{r} + u_r \cdot x_i \cdot \left(\frac{-1}{2r^3} \cdot 2x_i \right)$$

$$= u_{rr} \cdot \frac{(x_i)^2}{r^2} + u_r \cdot \frac{1}{r} - u_r \cdot \frac{(x_i)^2}{r^3}.$$

Now, summing over all indices, we get:

$$\Delta u = \sum_{i=1}^{n} \left[u_{rr} \cdot \frac{(x_i)^2}{r^2} + u_r \cdot \frac{1}{r} - u_r \cdot \frac{(x_i)^2}{r^3} \right]$$

$$= \sum_{i=1}^{n} \left[u_{rr} \cdot \frac{(x_i)^2}{r^2} \right] + \sum_{i=1}^{n} \left[u_r \cdot \frac{1}{r} \right] - \sum_{i=1}^{n} \left[u_r \cdot \frac{(x_i)^2}{r^3} \right]$$

$$= u_{rr} + u_r \cdot \frac{n}{r} - u_r \cdot \frac{1}{r}$$

$$\therefore \quad \Delta u(r) = u_{rr} + \left(\frac{n-1}{r} \right) u_r.$$

- -

(Continues)

Solution Technique (Continued):

The problem boils down to solving the following ODE then:

$$u_{rr} + \left(\frac{n-1}{r}\right)u_r = 0$$

- -

Note that if $u_r = 0$ then the solution is just a constant function. This option is encompassed in later results (i.e. $n = 1$ case with $C_1 = 0$). So, let us assume $u_r \neq 0$. Then we can rearrange and divide to get:

$$\frac{1}{u_r} \cdot u_{rr} = \frac{(1-n)}{r}.$$

$$\implies \partial_r(ln|u_r|) = \frac{1-n}{r}$$

$$\implies ln|u_r| = (1-n)ln|r| + C\ .$$

Upon integrating, we obtained the constant C. Let's use a rule of logarithms to bring the $(1-n)$ inside the log and then exponentiate the equation to rid of the logs and apply an exponent rule ($e^{x+y} = e^x \cdot e^y$) (#algebra2):

$$|u_r| = e^C|r^{1-n}|.$$

We may remove absolute value with a plus or minus out front, but we are going to relabel the constant:

$$A := \pm e^C$$

Hence:

$$u_r = Ar^{1-n}.$$

In the last step, we would like to just apply the Reverse Power Rule when integrating, but this gives us trouble when $1 - n = 0, -1$ that is, when $n = 1, 2$. Respectively in these two cases, we get:

$$u(r) = A\int r^0 dr = Ar + C_2$$

and $u(r) = A\int r^{-1}dr = Aln|r| + C_2$. The final case is clear then letting $N := (1-n) \geq 1$:

$$u(r) = A\int r^N dr = \frac{A}{N+1}r^{N+1} + C_2 = \frac{A}{2-n}r^{2-n} + C_2.$$

Renaming the constants one more time gives the result. ■

REALITY CHECK:

Let's make sure the solution obtained actually satisfies the PDE!

$$u(x) := \begin{cases} C_1 r + C_2 & \text{if } n = 1 \\ C_1 ln|r| + C_2 & \text{if } n = 2 \\ C_1 r^{2-n} + C_2 & \text{if } n \geq 3 \end{cases}$$

vs.

$$\Delta u = 0$$

Case (n=1):

$\partial_i u = C_1 \partial_i r = C_1 \frac{x_i}{r}$ and $\partial_{ii} u = C_1 \left(\frac{1}{r} + x_i \cdot \left(\frac{-x_i}{r^3} \right) \right)$.

$$\implies \Delta u = C_1 \cdot \frac{n}{r} - C_1 \sum_{i=1}^{n} \frac{(x_i)^2}{r^3} = 0 \;. \; \square$$

- -

Case (n=2):

$\frac{1}{C_1} \partial_i u = \frac{1}{r} \cdot \frac{x_i}{r}$ and $\frac{1}{C_1} \partial_{ii} u = \frac{1}{r^2} + x_i \cdot \frac{-2}{r^3} \cdot \frac{x_i}{r} = \frac{1}{r^2} - \frac{2}{r^4} x_i$.

$$\implies \frac{1}{C_1} \Delta u = \frac{n}{r^2} - \frac{2}{r^2} = (n-2)\frac{1}{r^2} = 0. \; \square$$

- -

Case ($n \geq 3$):

$\frac{1}{C_1} \partial_i u = (2-n) r^{2-n-1} \cdot \frac{x_i}{r} = (2-n) r^{-n} x_i$ and
$\frac{1}{(2-n)*C_1} \partial_{ii} u = r^{-n} + x_i \cdot (-n) r^{-n-1} \cdot \frac{x_i}{r} = r^{-n} + (-n) r^{-n-2} \cdot (x_i)^2$.

$$\implies \frac{1}{(2-n)C_1} \Delta u = n \cdot r^{-n} + (-n) r^{-n} = 0.$$

■

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3. (Homogeneous) Heat/Diffusion Equation (pgs.44-65 [5])

Domain and Unknown Function:

$$U_{open} \subseteq \mathbb{R}^n \times (0, \infty) \qquad \text{and} \qquad u : \bar{U} \to \mathbb{R}$$

Equation:

$$u_t - \Delta u = 0 \qquad \text{for } (x, t) \in U$$

Constraints:

$$u = g \qquad \text{for } (x, 0) \in \partial U$$

where $g : \partial U \to \mathbb{R}$ is a fixed known function on the boundary.

Solution:

$$u((x,t)) := \frac{A}{t^{n/2}} e^{\frac{-|x|_2^2}{4t}}$$

for $(x, t) \in U$. Boundary condition not yet met.

Solution Technique:

Apply theory from Section II.2.1-3.

(Continues)

<u>Solution Technique:</u> (I stress that this is adapted from Evan's text! Also to be clear, I developed a convention that $\partial_i u =: u_i$ etc. but x_i is just a coordinate. Should be clear from context hopefully.)

Suppose the solution has the form:

$$u(x,t) = t^{-\alpha} v(t^{-\beta} x).$$

- -

Compute using the Chain Rule, Product Rule, and Power Rule where appropriate:

$$\partial_t u = -\alpha t^{-\alpha-1} v(t^{-\beta}x) + t^{-\alpha} \cdot \sum_{i=1}^{n} v_i \cdot (-\beta t^{-\beta-1} x_i)$$

$$= -\alpha t^{-\alpha-1} v(t^{-\beta}x) - \beta t^{-\alpha-1} \cdot \sum_{i=1}^{n} (t^{-\beta} x_i) \cdot v_i$$

$$\partial_t u = -\alpha t^{-\alpha-1} v(t^{-\beta}x) - \beta t^{-\alpha-1} \cdot \langle t^{-\beta}x, Dv \rangle$$

- -

Similarly,

$$\partial_i u = t^{-\alpha} \partial_i v(t^{-\beta}x) = t^{-\alpha} \sum_{j=1}^{n} v_j \cdot t^{-\beta} \delta_j^i = t^{-\alpha-\beta} v_i$$

$$\implies \partial_{ii} u = t^{-\alpha-\beta} v_{ii} \cdot (t^{-\beta}) = t^{-\alpha-2\beta} v_{ii}$$

$$\implies \Delta u = t^{-\alpha-2\beta} \Delta v$$

- -

Combining these results we get our PDE becomes:

$$u_t - \Delta u = -\alpha t^{-\alpha-1} v(t^{-\beta}x) - \beta t^{-\alpha-1} \cdot \langle t^{-\beta}x, Dv \rangle - t^{-\alpha-2\beta} \Delta v = 0.$$

Dividing out the negative, setting $\beta := 1/2$, factoring out the common term and defining $y := t^{-\beta} x$, we get:

$$t^{-\alpha-1} \left[\alpha v(y) - \frac{1}{2} \langle y, Dv(y) \rangle - \Delta v(y) \right] = 0$$

But $t^{-\alpha-1} \neq 0$ for any finite t, so we conclude:

$$\alpha v(y) - \frac{1}{2} \langle y, Dv(y) \rangle - \Delta v(y) = 0$$

This is still a PDE, but at least the time parameter is gone. Notice it is radially symmetric in the spatial variables.

(Continues)

Solution Technique (Continued):

We have a new PDE that exhibits a radial symmetry:

$$\alpha v(y) - \frac{1}{2}\langle y, Dv(y)\rangle - \Delta v(y) = 0.$$

As in Laplace, we define $r := |y|_2$ and assume $v = v(r)$.

- -

Then pulling from previous work, we compute:

$$\partial_i v(r) = v_r \cdot \frac{y_i}{r}$$

and hence:

$$Dv(r) = \frac{v_r}{r} y \qquad \text{and} \qquad \Delta v(r) = v_{rr} + (\frac{n-1}{r})v_r$$

- -

Substituting into the above yields:

$$\alpha v(r) - \frac{1}{2}\langle y, \frac{v_r}{r}y\rangle - v_{rr} - (\frac{n-1}{r})v_r = 0$$

But real inner products are bilinear, so together with $r^2 = \langle y, y\rangle$, this gives:

$$\alpha v(r) - \frac{r}{2}v_r - v_{rr} - \left(\frac{n-1}{r}\right)v_r = 0$$

Rearranging:

$$v_{rr} + \left(\frac{n-1}{r} + \frac{r}{2}\right)v_r - \alpha v = 0.$$

- -

Now the trick is here to use *reverse product rule* twice (or *integrating factor* technique?) and select α so that things work. Observe that:

$$\partial_r\big(r^{n-1}v_r\big) = (n-1)r^{n-2}v_r + r^{n-1}v_{rr} \qquad \text{and} \qquad \partial_r\big(r^n v\big) = nr^{n-1}v + r^n v_r.$$

Then we get:

$$\partial_r\big(r^{n-1}v_r\big) + \frac{1}{2}\partial_r\big(r^n v\big) = r^{n-1}v_{rr} + (n-1)r^{n-2}v_r + \frac{r^n}{2}v_r + \frac{nr^{n-1}}{2}v.$$

$$= r^{n-1}\left[v_{rr} + \left(\frac{n-1}{r} + \frac{r}{2}\right)v_r - \frac{n}{2}v\right]$$

Which suggests we set $\alpha := n/2$ in order to collapse the equation into:

$$\partial_r\big(r^{n-1}v_r\big) + \frac{1}{2}\partial_r\big(r^n v\big) = 0$$

Solution Technique (Continued (2)):

We can directly integrate the above with respect to r:

$$r^{n-1}v_r + \frac{1}{2}r^n v = C_1$$

for some integration constant, C_1. We are interested in finding "a" solution, so assume we have $C_1 = 0$ and $v \neq 0$. Then we can rearrange:

$$\frac{v_r}{v} = -\frac{r}{2}$$

And the usual logarithm trick gives:

$$ln|v| = -\frac{1}{4}r^2 + C_2$$

Then exponentiating and defining $A := \pm e^{C_2}$ as usual:

$$v(r) = Ae^{-\frac{1}{4}r^2}$$

Now recall that originally our solution was of the form:

$$u(x,t) = t^{-\alpha}v(t^{-\beta}x)$$

But throughout we set $\beta = \frac{1}{2}$, $r = |y| = |t^{-\beta}x|$ and $\alpha = \frac{n}{2}$. Thus, finally we arrive at:

$$u(x,t) = t^{-n/2}Ae^{-\frac{1}{4}|t^{-1/2}x|^2} = \frac{A}{t^{n/2}}e^{\frac{-|x|_2^2}{4t}} \qquad \blacksquare$$

REALITY CHECK:

Let's make sure the solution obtained actually satisfies the PDE!

$$\boldsymbol{u((x,t)) = \frac{A}{t^{n/2}} e^{\frac{-|x|_2^2}{4t}}}$$

vs.

$$\boldsymbol{u_t - \Delta u = 0}$$

$$\begin{aligned}\boldsymbol{\partial_t u} &\boldsymbol{= (-n/2)At^{-n/2-1} \cdot e^{\frac{-|x|_2^2}{4t}} + At^{-n/2} e^{\frac{-|x|_2^2}{4t}} \cdot \frac{|x|^2}{4t^2}} \\ &\boldsymbol{= \frac{A}{t^{n/2}} e^{\frac{-|x|_2^2}{4t}} \cdot \left(\frac{|x|^2}{4t^2} - \frac{n}{2t} \right)} \\ &\boldsymbol{= u\left(\frac{|x|^2}{4t^2} - \frac{n}{2t} \right)}\end{aligned}$$

- -

$$\boldsymbol{\partial_i u = u \cdot \left(\frac{-1}{4t} \cdot 2x_i \right) = u \cdot \left(\frac{-x_i}{2t} \right)}$$

$$\begin{aligned}\boldsymbol{\implies \partial_{ii} u} &\boldsymbol{= \partial_i(u) \cdot \left(\frac{-x_i}{2t} \right) + u \cdot \partial_i \left(\frac{-x_i}{2t} \right)} \\ &\boldsymbol{= u \cdot \left(\frac{x_i^2}{4t^2} \right) - u \cdot \frac{1}{2t}} \\ &\boldsymbol{= u\left(\frac{x_i^2}{4t^2} - \frac{1}{2t} \right)}\end{aligned}$$

$$\boldsymbol{\implies \Delta u = u\left(\frac{|x|^2}{4t^2} - \frac{n}{2t} \right)}$$

and the result is clear. ■

---

4. (Homogeneous) Wave Equation (pgs.65-84 [5])

Domain and Unknown Function:

...

Equation:

...

Constraints:

...

...

Solution:

...

Solution Technique:

... Next page.

(Continues)

Solution Technique:

...

REALITY CHECK:

Let's make sure the solution obtained actually satisfies the PDE!

$$\ldots$$

vs.

$$\ldots$$

...

---

5. PLANAR WAVE SOLUTIONS - Collection of PDE's (§4.2 [5])

> Includes:
(1) **Heat** Eq. (as in earlier entry), (2) **Wave**, (3) **Klein-Gordon**, (4) **Schrodinger**, and (5) **Airy**.

Since we are doing multiple at once, this article will be a little different in format.
All homogeneous and without auxiliary data. Just solutions at the moment.

Domain and Unknown Function:

$$U_{open} \subseteq \mathbb{R}^n \times (0, \infty) \qquad \text{and} \qquad u : \bar{U} \to \mathbb{R}$$

Separate Equation(s):

(1) $u_t - \Delta u = 0$

(2) $u_{tt} - \Delta u = 0$

(3) $u_{tt} - \Delta u + m^2 u = 0$

(4) $iu_t + \Delta u = 0$

(5) $u_t + u_{xxx} = 0 \quad (n = 1)$

For any $y \in \mathbb{R}^n$:

(1) $u(x,t) = e^{-|y|_2^2 t} e^{iy\cdot x}$.

(2) $u(x,t) = e^{\pm |y| t} e^{iy\cdot x}$

(3) $u(x,t) = e^{\pm\sqrt{|y|^2+m^2}t} e^{iy\cdot x}$

(4) $u(x,t) = e^{i(y\cdot x - |y|^2 t)}$

(5) $u(x,t) = e^{i(yx + y^3 t)}$

Solution Technique:

Apply Undetermined Coefficients (See II.2.3), using *planar waves* (model function):

$$u(x,t) = e^{i(y\cdot x - \sigma t)},$$

where $y \in \mathbb{R}^n$ and we allow $\sigma \in \mathbb{C}$.

(Continues)

Solution Technique: These are fairly easy.

(HEAT: $u_t - \Delta u = 0$)

1.) Let $u(x,t) = e^{i(y\cdot x - \sigma t)}$ for free $y \in \mathbb{R}^n$ and $\sigma \in \mathbb{C}$, then we get:

$$\partial_t u = -i\sigma e^{i(y\cdot x-\sigma t)} = -i\sigma u(x,t)$$

and

$$\partial_j u = iy^j e^{i(y\cdot x-\sigma t)}$$
$$\implies \partial_{jj} u = i^2 (y^j)^2 e^{i(y\cdot x-\sigma t)} = -(y^j)^2 u(x,t)$$
$$\implies u_t - \Delta u = \left[-i\sigma + \sum_{j=1}^{n} (y^j)^2\right] u(x,t) = \left[-i\sigma + |y|_2^2\right] u(x,t)\ .$$

This equals zero precisely when:

$$\sigma = \frac{1}{i}|y|^2 = -i|y|_2^2.$$

Hence:

$$u(x,t) = e^{i(y\cdot x + i|y|_2^2 t)} = e^{iy\cdot x - |y|_2^2 t} = e^{-|y|_2^2 t} e^{iy\cdot x}$$

□

(Wave: $u_{tt} - \Delta u = 0$)

2.) Let $u(x,t) = e^{i(y\cdot x - \sigma t)}$ as before. Then:

$$\partial_t u = -i\sigma u$$

$$\partial_{tt} u = -\sigma^2 u$$

and

$$\Delta u = |y|^2 u$$
$$\implies u_{tt} - \Delta u = (-\sigma^2 - |y|^2) u = 0$$
$$\implies \sigma^2 = -|y|^2$$
$$\implies \sigma = \pm i|y|$$

Hence:

$$u(x,t) = e^{i(y\cdot x \mp i|y|t)} = e^{\pm|y|t} e^{iy\cdot x}$$

□

(3-5.) [Exercise: The rest are for you to check!] ■

REALITY CHECK:

The proof is in the puddin-pop this time! See above. ■

(End of Entry 5)

6. (Homogeneous) Conservation Laws (p.113 [5])

Domain and Unknown Function:

$$U_{open} \subseteq \mathbb{R}^n \times (0, \infty) \qquad \text{and} \qquad u : \bar{U} \to \mathbb{R}$$

Equation:

$$u_t + H(x, Du) = 0 \qquad \text{for } (x,t) \in U$$

Constraints:

$$u = g \qquad \text{for } (x, 0) \in \partial U$$

where $g : \partial U \to \mathbb{R}$ is a fixed known function on the boundary and $H : \mathbb{R}^{2n} \times \mathbb{R}$ is also fixed.

Solution:

...

Given by solving the two systems from the theory (when possible).
Solution varies with instances of $H(x, Du)$.

Solution Technique:

Apply **Algorithm II.1.2**. This is done on the next page. Some of the work has been done already in **Technical Results Entry 4**.

(Continues)

Solution Technique:

(1)-(3) Have been taken care of in the aforementioned Results Entry. It should be noted that we incorporated time as a variable and had to rename variables from the algorithm to be consistent. We have:

$$F(y(s), z(s), q(s)) := q^{n+1}(s) + H(x(s), p(s)).$$

4.) To solve the first system, we need to compute partials of F. One of these was taken care of already (namely: $\partial_z F = 0$).

Next, compute:

$$\partial_{q^i} F = \partial_{q^i}(q^{n+1}) + \partial_{q^i} H(x,p) = \begin{cases} \partial_{p^i} H & \text{when } i \neq n+1 \\ 1 & \text{when } i = n+1 \end{cases}.$$

Lastly (in disagreement with the text), we compute:

$$\begin{aligned} \partial_{y^i} F &= \partial_{y^i}(q^{n+1}) + \partial_{y^i}\big(H(x,p)\big) \\ &= \partial_{y^i}\partial_{y^{n+1}} u(y) + \left(\partial_{y^i} H + \sum_{j=1}^{n} \partial_{q^j} H \cdot \partial_{y^i}\partial_{y^j} u(y)\right) \\ &= \partial_{y^{n+1}} q^i + \partial_{y^i} H + \sum_{j=1}^{n} \partial_{q^j} H \cdot \partial_{y^j} q^i \end{aligned}$$

In this last step, we commute the partials (as we are in flat space), as to prioritize q^i in the expression. It should also be noted that $\partial_{y^i} H$ represents the partial w.r.t. the i^{th} variable of H (think: $H(w_1, ..., w_{2n})$). Now, there are two possibilities for i again here:

$$\partial_{y^i} F = \begin{cases} \partial_{y^{n+1}} q^i + \partial_{y^i} H + \sum_{j=1}^{n} \partial_{p^j} H(q^i) \cdot \partial_{y^j} q^i & \text{when } i \neq n+1 \\ \partial_{y^{n+1}} q^{n+1} + \sum_{j=1}^{n} \partial_{p^j} H \cdot \partial_{y^j} q^{n+1} & \text{when } i = n+1 \end{cases}$$

Plugging these results into the first system in the algorithm we get...

Solution Technique (Continued):

$$\dot{y}^i = \begin{cases} \partial_{p^i} H & \text{when } i \neq n+1 \\ 1 & \text{when } i = n+1 \end{cases}$$

$$\dot{z} = \sum_{j=1}^{n+1} \left(\begin{cases} \partial_{p^j} H & \text{when } j \neq n+1 \\ 1 & \text{when } j = n+1 \end{cases} \right) \cdot q^j$$

$$\dot{q}^i = -\begin{cases} \partial_{y^{n+1}} q^i + \partial_{y^i} H + \sum_{j=1}^{n} \partial_{p^j} H(q^i) \cdot \partial_{y^j} q^i & \text{when } i \neq n+1 \\ \partial_{y^{n+1}} q^{n+1} + \sum_{j=1}^{n} \partial_{p^j} H \cdot \partial_{y^j} q^{n+1} & \text{when } i = n+1 \end{cases}$$

To actually solve this system, we need an instance for H that admits a solution.

To obtain such an H that is Physically derivable, interpret-able in our notation, and simple enough to admit a solution is more down the rabbit hole than I'm willing to go here (especially since we have a disagreement with the text for the expression of $\partial_{y^i} F$. So the remainder is left as an [Exercise: Finish the fight!]

REALITY CHECK:

Let's make sure the solution obtained actually satisfies the PDE!

$$\ldots$$

vs.

$$u_t + H(x, Du) = 0$$

...

---

7. (Homogeneous) System from Stoke's Theorem (See Entry 3)

Domain and Unknown Function:

$$U_{open} \subseteq \mathbb{R}^3 \qquad \text{and} \qquad \omega \in \Gamma^2(T^*(TU))$$

with boundary $\partial U = C$ (a closed loop). Assume U has no holes so that $d\omega$ is exact.

Equations:

$$\text{Given: } d\omega := (1+x^2)f(x)dydz - 2xyf(x)dzdx - 3zdxdy$$

$$f(x) = 3tan^{-1}(x)$$

- -

$$(i)\ \partial_y\omega_3 - \partial_z\omega_2 = (1+x^2)f(x)$$

$$(ii)\ \partial_z\omega_1 - \partial_x\omega_3 = -2xyf(x)$$

$$(iii)\ \partial_x\omega_2 - \partial_y\omega_1 = -3z$$

Solution:

$$\omega = (-6xyz \cdot tan^{-1}(x))dx + (-3z(1+x^2) \cdot tan^{-1}(x))dy$$

Solution Technique:

Just assuming what we can get away with setting equal to zero some of the components and seeing what works. This is like forcing the system to be diagonal or traingular.

(Continues)

REALITY CHECK:

Let's make sure the solution obtained actually satisfies the system!

$$\omega = (-6xyz \cdot tan^{-1}(x))dx + (-3z(1+x^2) \cdot tan^{-1}(x))dy$$
$$f(x) = 3tan^{-1}(x)$$

vs.

$$(i)\ \partial_y \omega_3 - \partial_z \omega_2 = (1+x^2)f(x)$$
$$(ii)\ \partial_z \omega_1 - \partial_x \omega_3 = -2xyf(x)$$
$$(iii)\ \partial_x \omega_2 - \partial_y \omega_1 = -3z$$

Define:

$$\omega_1 := -6xyz \cdot tan^{-1}(x),$$
$$\omega_2 := -3z(1+x^2) \cdot tan^{-1}(x), \quad \text{and}$$
$$\omega_3 = 0.$$

Then:

$$\begin{bmatrix} \partial_x \omega_1 = -6yz[tan^{-1}(x) + \frac{x}{1+x^2}] & \partial_y \omega_1 = -6xz \cdot tan^{-1}(x) & \partial_z \omega_1 = -6xy \cdot tan^{-1}(x) \\ \partial_x \omega_2 = -3z[2x \cdot tan^{-1}(x) + 1] & \partial_y \omega_2 = 0 & \partial_z \omega_2 = -3(1+x^2) \cdot tan^{-1}(x) \\ \partial_x \omega_3 = 0 & \partial_y \omega_3 = 0 & \partial_z \omega_3 = 0 \end{bmatrix}.$$

By comparing entries of the above matrix, one can see the result. ■

Breathe!

II: General Techniques for Classes of Equations

As the structure of the problem varies, so too must the method of attack.

As a quick survey of what I've witnessed in the literature, for ODE's, we have the brute force method (algebraically manipulate equation and integrate, when possible); One can factor polynomial operators and find solution to linear factors (taking extra steps for repeated roots) and applying the Superposition Principle; Use the multiplier method to change to reverse product rule problem; Guess solutions (such as complex exponentials) with Undetermined Coefficients (apply auxiliary data to solve for these later); In this same avenue, guess Series Solutions and solve corresponding Recurrence Equations; For systems, either diagonalize or triangularize the matrix, brute force when possible; Using Fundamental Solutions to apply Variation of Parameters Method, Laplace and other Integral Transform Methods, etc.

For PDE's, I've seen Separation of Variables (featuring Fourier Series); the Method of Characteristics, Undetermined Coefficients, and some Averaging over metric balls (as in solution to the Wave Eq.). For systems, zeroing out components helps get down to a more fundamental solution. Numerically, I'll say Iterative Procedures exist, using the PDE expression itself–here convergence and error tolerance are crucial aspects of study.

In this section, I will present some of the methods pointed to from within the Solutions Catalog (i.e. Section I).

Sub-Table of Contents:

⟨⟨ Foliating the Graph ⟩⟩

II.1.1: Method of Characteristics
[First Order, Linear, Non-homogeneous]

$$F(x, u, \partial_1, ..., \partial_n) := \sum_{i=1}^{n} \lambda_i \partial_i u - \lambda_{n+1} = 0,$$

The references for this section are: [5, 11, 7, 6]. Suppose throughout this section that continuous means continuous in each variable separately.

• Def: Briefly, for a given surface (or manifold) S, the **tangent bundle**, $T(S)$, is the disjoint union of **tangent spaces**, $T_p(S)$, for each point $p \in S$. A **tangent vector** $v_p \in T_p(S)$ can be defined by the velocity vector of any curve through that point.

• **Prop:** Suppose a continuous $f : \mathbb{R}^N \to \mathbb{R}$ is given and define a **level surface** as $S_c := f^{-1}(\{c\})$, for $c \in \mathbb{R}$. Then:

$$\nabla f\big|_{S_c} \perp T(S_c).$$

That is, the *gradient field* is orthogonal to the *tangent bundle* over the level surface.

Proof: Let $p \in S_c$ be arbitrary. Then by definition of **tangent vector on the level surface**, $v_p \in T_p(S_c)$, there exists a curve:

$$\gamma : \mathbb{R} \to S_c$$
$$t \mapsto \left(\gamma^1(t), ..., \gamma^N(t)\right)$$

such that $\gamma(0) = p$, $\quad v_p = \partial_t \gamma(t)\big|_{t=0}$, $\quad$ and $\quad \forall t,\ (f \circ \gamma)(t) = c$.

Now, take the *standard inner product*:

$$\langle\ \nabla f\big|_p, v_p\rangle := \langle\ \sum_{i=1}^{N} \partial_i f\big|_p \partial_i,\ \sum_{j=1}^{N} \partial_t \gamma^j(t)\big|_{t=0} \partial_j\ \rangle$$

$$= \sum_{k=1}^{N} \left(\partial_k f\big|_p \cdot \partial_t \gamma^k(t)\big|_{t=0}\right) = \partial_t (f \circ \gamma)(t)\big|_{t=0}.$$

Where the last equality is by the chain rule (see Entry 1). Now, we notice that $\partial_t (f \circ \gamma) \equiv 0$. This gives us the result by arbitrariness of $v_p \in T_p(S_c)$ and then arbitrariness of $p \in S_c$. ■

This clever use of the chain rule was inspired by (pg.917+ [7]).

(Continues)

• **Prop:** Let $u : \mathbb{R}^n \to \mathbb{R}$ be continuous. Then we have:

$$< \nabla u, -1 > \big|_{\Gamma u} \perp T(\Gamma u)\ .$$

That is, the vector consisting of the gradient and -1 in the last slot is orthogonal to the graph of u.

Proof: We may define an extended function whose zero level set is the desired graph:

$$\widetilde{u} : \mathbb{R}^{n+1} \to \mathbb{R};$$

$$(x_1, ..., x_n, x_{n+1}) \mapsto u(x_1, ..., x_n) - x_{n+1},$$

We apply the previous proposition with $c := 0$ and $N := n+1$ to arrive at:

$$\nabla \widetilde{u}\big|_{S_0} \perp T(S_0).$$

Then replace: $S_0 = \Gamma u$ and $\nabla \widetilde{u} := \sum_{i=1}^{n+1} \partial_i \widetilde{u} \partial_i = \sum_{i=1}^{n} \partial_i u \partial_i - 1 \cdot \partial_{n+1} =:< \nabla u, -1 >$. ■

Recall our PDE takes the form:

$$F(x, u, \partial_1, ..., \partial_n) := \sum_{i=1}^{n} \lambda_i \partial_i u - \lambda_{n+1} = 0,$$

where the coefficient functions are only spatially dependent $\lambda_i = \lambda_i(x)$.

- -

This can be factored into an inner product form:

$$\langle \sum_{i=1}^{n+1} \lambda_i \partial_i, \ \ < \nabla u, -1 > \rangle = 0.$$

where the vector fields defined on the graph (but evaluated under it), can be given names:

$$\Lambda(x) := \sum_{i=1}^{n+1} \lambda_i(x) \partial_i \qquad \text{and} \qquad \eta(x) :=< \nabla u(x), -1 >\ .$$

Rewriting:

$$\langle \Lambda, \eta \rangle = 0.$$

The recent proposition gives us that η is (orthogonal) to the graph of u. Since Λ is ortho to η, we deduce that Λ is tangent to Γu! Otherwise written in abusive *section* notation:

$$\eta \in \Gamma\big(T(\Gamma u)^{\perp}\big)$$

$$\Lambda \in \Gamma(T(\Gamma u))$$

(Next Page)

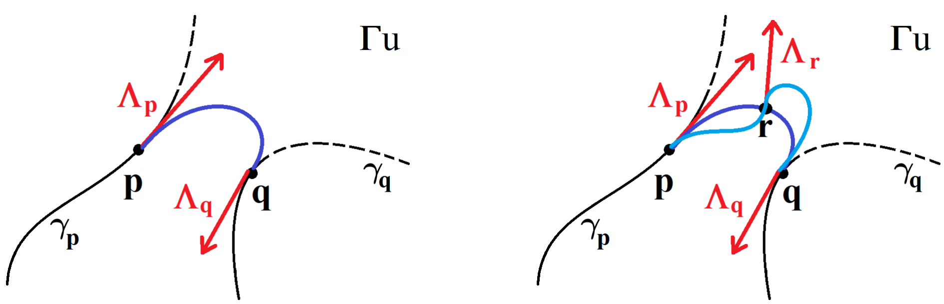
Restating the tangency of $\boldsymbol{\Lambda}$ with the definition of tangent vector applied:

• **Prop:** For continuous $\boldsymbol{u}:\mathbb{R}^n\to\mathbb{R}$, solving our PDE, there exist differentiable curves through each point $\left\{\gamma_p(t)\right\}_{p\in\Gamma u}$ in the graph whose velocities at those points are the vectors, $\left\{\boldsymbol{\Lambda}(\boldsymbol{x})\big|_p\right\}_{p\in\Gamma u}$, from the associated field $\boldsymbol{\Lambda}$. ■

We now have curves through each point satisfying the velocity condition (only at their base point). We need them to satisfy the velocity condition at every point (to make them **integral curves**). The next proposition accomplishes this.

• **Prop:** Taking the previous proposition as hypothesis, we may construct any integral curve of $\boldsymbol{\Lambda}$ from the family $\left\{\gamma_p\right\}_{p\in\Gamma u}$.

Proof: Consider any two points $\boldsymbol{p},\boldsymbol{q}\in\Gamma\boldsymbol{u}$ and their associated curves γ_p,γ_q in the family. We may construct a new curve $\gamma_{p,q}$ passing through both points with matching velocities to γ_p and γ_q by truncating both curves and appending a smooth curve (of our choice in $\Gamma\boldsymbol{u}$) as suggested by the figure (left).



Consider now cutting this appended curve at a midpoint and replacing by two new curves with this same technique (right figure). Iterate on both sides of the midpoint.

If at any stage of this process if we **break differentiability** of the constructed curve:
Truncate the future of the curve past the break point. Pick a new point, $\boldsymbol{q'}$, in an epsilon neighborhood of the broken point to connect to. Continue.

If the above process **terminates**, extend the farthest future endpoint by the same epsilon neighborhood point selection.

Once the future is completed to the extent possible (finite or infinite), repeat in past direction with epsilon neighborhood point selection: $\boldsymbol{q'}\in\boldsymbol{B_\epsilon(p)}$, choose curve in between as in the initial step. Repeat. Carrying out this process to completion constructs the integral curve. ■

★ **Theorem:** For a first order, linear, (non-homogenous) PDE the graph of a continuous solution, $\boldsymbol{u}: \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}$ is equal to the union of integral curves of the tangential vector field.

Proof: Given $\boldsymbol{p} \in \boldsymbol{\Gamma u}$, we have an integral curve through $\boldsymbol{p}$ since we have shown $\boldsymbol{\Lambda}$ is tangent at every point. Arbitrariness of $\boldsymbol{p}$ gives one containment:

$$\boldsymbol{\Gamma u} \subseteq \bigcup_{p \in \boldsymbol{\Gamma u}} \boldsymbol{Im}(\gamma_{\boldsymbol{p}}(\boldsymbol{t})).$$

Now suppose $\boldsymbol{q} = \gamma_{\boldsymbol{p}}(\boldsymbol{t})$ for some integral curve passing through a point $\boldsymbol{p}$. By the previous proposition, we have shown that $\gamma_{\boldsymbol{p}}(\boldsymbol{t})$ consists of concatenated curves lying entirely within $\boldsymbol{\Gamma u}$. Hence $\boldsymbol{q} \in \boldsymbol{\Gamma u}$. Arbitrariness $\boldsymbol{q} \in \bigcup_{p \in \boldsymbol{\Gamma u}} \boldsymbol{Im}(\gamma_{\boldsymbol{p}}(\boldsymbol{t}))$ gives the reverse containment. Hence equality. ■

On the next page, we use the above Theorem to create an algorithm for solving the PDE.

ALGORITHM: "Method of Characteristics a.k.a. of Integral Curves"

1.) Identify the PDE as first order, linear.

2.) Factor it into the form $\langle \Lambda, \eta \rangle = 0$ (See above for details).

3.) Pick a point $x \in \partial U$, then $p = (x, u(x)) \in \partial \Gamma u$. If Λ permits, solve the following:

$$\left\{ \quad \dot{\gamma}_p^i(t) = \Lambda^i(\widetilde{\gamma}(t))\big|_{\gamma_p(t)} \quad \right\}_{i \in \{1,\dots,n+1\}},$$

with appended initial data $\gamma_p(0) = p$.

4.) Realize $(x, u(x)) = p = \gamma_p(0) = \left(\gamma_p^1(0), \dots, \gamma_p^n(0), \gamma_p^{n+1}(0)\right)$ implies :

$$u(x) = \gamma_p^{n+1}(0).$$

Note: Integration of the system in (3) gives us $n+1$ new undetermined coefficients (a.k.a. $+C^i$). The initial data adds (n) more equations to help us solve for these unknowns. So, there is one piece of data missing and hence there should be another condition needed to fully solve.

5.) The **boundary condition** states:

$$u(x) = g(x) \text{ for all } x \in \partial U, \text{ given some function } g : \partial U \to \mathbb{R}.$$

From this, we deduce that on the boundary (as we are):

$$g(x) = \gamma_p^{n+1}(0).$$

And this provides the final equation to solve for the remaining coefficient.

6.) Upon completion of (1)-(5), the expression for $\gamma_p(t)$ will be known explicitly.

By the above Theorem, this is a union component of the graph of the solution. So, we may observe the values, $u(x') = \gamma_p^{n+1}(t')$ of the solution for $x' \in dom(u)$, by finding the corresponding parameter value $t' \in \mathbb{R}$ (which is basically the address of the point $(x', u(x'))$ on the image of $\gamma_p(t)$).

Unfortunately, finding t' depends on *local invertibility* of the projection:

$$\widetilde{\gamma_p} := proj_{\mathbb{R}^n}\left(\gamma : \mathbb{R} \to (\Gamma u)\right).$$

If *$\widetilde{\gamma}$ is locally invertible about the original boundary point $x \in \partial U$ and x' is in that neighborhood* then we have the *solution along a curve through p*:

$$\forall x' \in S \subseteq Im(\widetilde{\gamma}_p(t)), \qquad u(x') = \gamma_p^{n+1}\left(\widetilde{\gamma}_p^{-1}(x')\right).$$

7.) Use $x' = \widetilde{\gamma_p}(t')$ to eliminate x if it appears. [END ALGORITHM]

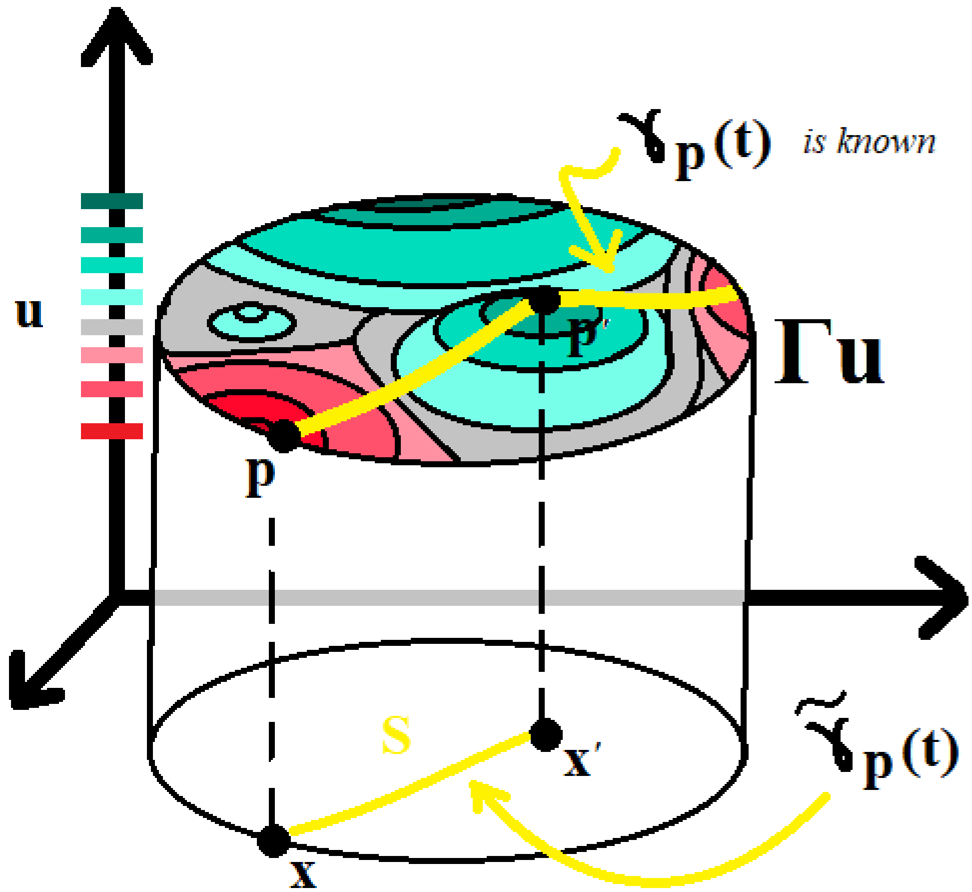


Figure: Context of Algorithm II.1.1

II.1b: Remarks on the Solution

Explicit Forms Require 2 More Conditions:

Reiterating, we have the restricted solution to the main PDE problem looks like:

$$\boldsymbol{u : U \subseteq \mathbb{R}^n \to \mathbb{R};}$$

$$\boldsymbol{u := \coprod_{x \in \partial U} u\big|_{S_x}},$$

where the union components are solutions parameterized over their projected integral curves, $\widetilde{\gamma}_p(t)$, through given boundary points, $p = (x, u(x)) \in \partial\Gamma u$, where:

$$u\big|_{S_x} : S_x \subseteq Im\big(\widetilde{\gamma}_{(x,u(x))}(s)\big) \hookrightarrow \mathbb{R}$$

$$u\big|_{S_x}(x') := \gamma_p^{n+1}(\widetilde{\gamma}_p^{-1}(x')).$$

To get invertibility of $\widetilde{\gamma}_p(t)$, we had to restrict to a subset $S_x := B_\epsilon(0) \cap dom\big(\widetilde{\gamma}_p(s)\big)$. This in effect allowed us to eliminate the address parameter, t'.

- -

A trick we've uncovered in some of the problems led us to conditionally go further and eliminate the boundary variable, $p = (x, u(x))$, leaving the solution completely determined by x' only. This is good since then the initial blue presentation of the solution is not lying.

$$U := \coprod_{x \in \partial U} S_x,$$

so we take $x' \in U$ and plug it in for the result $u(x')$.

The condition being that the expression for

$$x' = (x'_1, ..., x'_n) = \widetilde{\gamma}_p(t') = \big(\widetilde{\gamma}_p^1(t'), ..., \widetilde{\gamma}_p^n(t')\big)$$

must be linear in the boundary variable $x = (x_1, ..., x_n)$, in each component, so that we may solve for x in terms of x'. If this is not the case, apply other methods to solve.

On Regularity:

The non-homogeneous term, λ_{n+1} and the boundary function, g, show up in the last coordinate function of γ_p when we solve the system of ODE's. For u to be differentiable to any order then requires at least λ_{n+1} and g to both be differentiable of that order.

It is typically easier to judge regularity after you've gone through with the algorithm.

If λ_{n+1} or g are contrived to not be differentiable, the solution above is called a **weak solution**. This is relevant when we talk about computing the derivatives.

II.1: Method of Characteristics 2
[First Order, Non-Linear, Non-homogeneous]

$$F(x, u, \partial_1, ..., \partial_n) = F(x, u, D^1 u) = 0.$$

(Non-hom. is built in to non-linearity). The reference for this section is (p.96+ [5]).

Last time, we were able to take the form of the equation and factor it into an inner product of vector fields, which eventually led to the notion of the *graph of the solution* being *(locally) foliated* by integral curves of the vector field.

This time, we don't get such a convenient factorization of the equation. However, if we still assume the graph of the solution is locally foliated (by curves, $\gamma_p : I \subseteq \mathbb{R} \hookrightarrow \Gamma u$, related to the operator in some way), we can actually stumble across another system of ODE's, albeit different than before, but which will lead to the explicit solution.

If the graph is foliated, then the domain can be foliated by a projection of these curves down to last coordinate zero (recall the $\widetilde{\gamma}_p(t)$ from Algorithm II.1.1 figure). Let us now consider one such curve through a boundary point, $p \in \partial\Gamma u \subseteq \mathbb{R}^{n+1}$, and relabel:

$$x(s) := \widetilde{\gamma}_p(s) = (x^1(s), ..., x^n(s))$$

$$x_0 := x(0) := (p^1, ..., p^n).$$

(We replace t with s to avoid conflict when time is a variable. This should have been done before, but without experience how were we supposed to know!)

The next step is to restrict our master equation to this curve and manipulate it from there to get the system. Since we have:

$$F(x, u, Du) = 0$$

restricting to the parameterization looks like:

$$F\Big(x(s), u(x(s)), Du(x(s))\Big) = 0.$$

On the next page, we will be taking derivatives of the master equation with respect to each of its variables, so it is convenient to define:

$$z(s) := u(x(s)) \qquad \text{and} \qquad \big[q(s) := Du(x(s))\big] \leftrightarrow \big[q^i(s) := \partial_i\big(u(x(s))\big)\big]$$

In other words, we write: $F(x(s), z(s), q(s)) = 0$ so derivatives look like $\partial_{x^i(s)}$, $\partial_{z(s)}$, and $\partial_{q^i(s)}$.

Reiterating:

$$\boldsymbol{x(s)} = (\boldsymbol{x^1(s), ..., x^n(s)}) \qquad \text{with} \qquad \boldsymbol{x_0 = x(0) = \widetilde{p},}$$
$$\boldsymbol{z(s)} = \boldsymbol{u(x(s))} \qquad \text{and} \qquad \boldsymbol{q^j(s) = \partial_j(u(x(s)))}.$$

Since $\boldsymbol{z(s)}$ and $\boldsymbol{q^i(s)}$ have dependence on $\boldsymbol{x(s)}$, from the Chain Rule, we get (suppressing $\boldsymbol{s}$):

$$\begin{aligned}\boldsymbol{\partial_{x^i} F(x,z,q)} &= \sum_{j=1}^{n} \boldsymbol{\partial_{x^j} F(x,z,q) \cdot \delta_i^j} \\ &\quad + \boldsymbol{\partial_z F(x,z,q)} \cdot \left(\sum_{j=1}^{n} \boldsymbol{\partial_{x^j} u(x) \delta_i^j} \right) \\ &\quad + \sum_{j=1}^{n} \boldsymbol{\partial_{q^j} F(x,z,q)} \cdot \left(\sum_{k=1}^{n} \boldsymbol{\partial_{x^k x^j} u(x) \cdot \delta_i^k} \right)\end{aligned}$$

$$= \boldsymbol{\partial_{x^i} F(x,z,q) + \partial_z F(x,z,q) \cdot \partial_{x^i} u(x)} + \sum_{j=1}^{n} \boldsymbol{\partial_{q^j} F(x,z,q) \cdot \partial_{x^i x^j} u(x)} \ .$$

Recall: $\boldsymbol{\delta_b^a} = \begin{cases} \mathbf{1}, & \text{if } \boldsymbol{a = b} \\ \mathbf{0}, & \text{otherwise} \end{cases}$ *is the Kronecker Delta.*

Now, note that:

$$\Big(\boldsymbol{F(x,z,q) = 0}\Big) \implies \forall \boldsymbol{i} : \Big(\boldsymbol{\partial_{x^i} F(x,z,q) = \partial_{x^i}(0) = 0}\Big).$$

So the expressions above can be collected as follows (updating variable names):

(Next Page)

Continuing, we have our original PDE and the Chain Rule gave us a set of equations for each $\boldsymbol{\partial}_{\boldsymbol{x}^i}$:

$$\left\{ \sum_{j=1}^{n} \partial_{q^j} F(x, z, q) \cdot \partial_{x^i} q^j(x) = -\Big(\partial_{x^i} F(x, z, q) + \partial_z F(x, z, q) \cdot q^i(x) \Big) \right\}_{i \in \{1, \ldots, n\}}$$

Let us now observe some information provided by the parameter-derivative, $\boldsymbol{\partial}_{\boldsymbol{s}}$:

(the cases for the $\partial_s x^i(s)$'s yield nothing new),

$$\partial_s z(s) = \sum_{j=1}^{n} \partial_{x^j} z(s) \cdot \partial_s x^j(s) = \sum_{j=1}^{n} q^j(s) \cdot \partial_s x^j(s),$$

and

$$\left\{ \partial_s q^i(s) = \sum_{j=1}^{n} \partial_{x^j} q^i(s) \cdot \partial_s x^j(s) = \sum_{j=1}^{n} \partial_s x^j(s) \cdot \partial_{x^i} q^j(s) \right\}_{i \in \{1, \ldots, n\}}.$$

Written in this way, a possible connection between the *darkBlue* and *darkRed* equations presents itself. Note the slight of hand in the last equality: $\partial_{x^j} q^i = \partial_{x^i} q^j$, since in flat space mixed partials of $\boldsymbol{u}$ commute.

To connect the two equations would require:

$$\forall k \in \{1, \ldots, n\} : \quad \partial_{q^k} F(x, z, q)(s) = \partial_s x^k(s).$$

However, we do not know these equations hold apriori.

(Next Page)

If we conditionally continue with this assumption, the bridge is made and we gather the system of equations (from the previous page) over $\boldsymbol{i \in \{1, ..., n\}}$:

$$\begin{aligned} &\dot{x}^i = \partial_{q^i} F \\ \implies &\dot{q}^i = -(\partial_{x^i} F + \partial_z F \cdot q^i) \\ \implies &\dot{z} = \sum_{j=1}^{n} q^j \cdot \partial_{q^j} F \end{aligned}$$

where of course, we've shorthanded $\boldsymbol{\partial_s \varphi(s) =: \dot{\varphi}}$ and suppressed all arguments of functions for clarity. And applied the new assumption in the $\boldsymbol{\dot{z}}$ equation.

There are 2n+1 equations here and 2n+1 unknowns: $\boldsymbol{\{x^i, z, q^i\}}$. Provided we are able to solve this system (of ODE's), we have enough equations to do so.

The next step following the solution of this system is to determine the coefficients that appear via $\boldsymbol{s}$-integration. This is where the *auxiliary data* (here the boundary condition) comes into play. The analysis starts again on the...

(Next Page)

We have by this point, found expressions for all $x^i(s)$, $z(s)$, and $q^i(s)$ up to some s-integration-coefficients (there should be 2n+1 of them) [Exercise: Verify this with an explicit PDE problem]. To solve for these, we'll need 2n+1 equations again.

To start, we have from the very beginning:

$$x^i(0) = p^i$$

and with a boundary condition for "known" $g : \partial U \to \mathbb{R}$:

$$z(0) = u(x(0)) = g(x(0)).$$

Lastly, we impose the further *initial conditions*:

$$q^i(0) = \partial_{x^i} g(x(0))$$

which serve to complete the system. This however, requires g to be at least $C^1(\partial U)$.

Now that the theory is done. The algorithm is streamlined next:

ALGORITHM: "Nonlinear Method of Characteristics"

1.) Identify the PDE as first order, nonlinear with boundary condition:

$$F(x, u, D^1 u) = 0$$

$$u|_{\partial U} \equiv g,$$

where $g : \partial U \to \mathbb{R}$ is C^1.

2.) Select a point, $\widetilde{p} \in \partial U$ and suppose $\widetilde{\gamma}_p : I \subseteq \mathbb{R} \hookrightarrow U$ is an *injective* curve in the domain of u, going through $\widetilde{p}$. Define:

$$x(s) := \widetilde{\gamma}_p(s)$$

That is, $\forall i,\ x^i(s) = \widetilde{\gamma}^i(s)$. And suppose we reparameterize to make:

$$x_0 := x(0) := \widetilde{p}.$$

3.) Further define variables:

$$z(s) := u(x(s)) \qquad \text{and} \qquad q(s) := D_x u(x(s))$$

that is,

$$\forall i,\ q^i(s) = \partial_{x^i} u(x(s)).$$

We thus have:

$$F(x, z, q) = 0.$$

4.) Solve the system:

$$\dot{x}^i = \partial_{q^i} F$$

$$\dot{q}^i = -(\partial_{x^i} F + \partial_z F \cdot q^i)$$

$$\dot{z} = \sum_{j=1}^{n} q^j \cdot \partial_{q^j} F$$

for the expressions $\{x(s), z(s), q(s)\}$, provided the system is solvable.

(Continues)

Continuing:

5.) Following the solution to (4), solve for remaining integration constants with the system:

$$\boldsymbol{x}^i(0) = \widetilde{\boldsymbol{p}}^i$$

$$\boldsymbol{z}(0) = \boldsymbol{g}(\boldsymbol{x}(0))$$

$$\boldsymbol{q}^i(0) = \boldsymbol{\partial}_{\boldsymbol{x}^i}\boldsymbol{g}(\boldsymbol{x}(0)).$$

6.) $\boldsymbol{z}$ represents the solution along the curve $\boldsymbol{x}$. Possible inputs for the solution may be read off of the image of the curve. To the extent that the curve is injective, the expression for the boundary point may be eliminated (which is inherently desirable for "gluing" the solutions together for the global result).

[END ALGORITHM]

⟨⟨ Appealing to Symmetries ⟩⟩

II.2.1: Translations, Dilations, Reflections, and Rotations

This section was motivated by Ch.6, particularly (pgs.159-160 of [1]).

Consider $V := \mathbb{R}^n$ as a **vector space** with a **basis** $\boldsymbol{\beta}$. Then:

• <u>Def:</u> A **translation** is a map:

$$\boldsymbol{T : V \to V}$$

$$\boldsymbol{v \mapsto v + w}$$

for some vector $\boldsymbol{w} \in \boldsymbol{V}$. In coordinates this says:

$$\begin{bmatrix} \boldsymbol{v_1} \\ \vdots \\ \boldsymbol{v_n} \end{bmatrix} \mapsto \begin{bmatrix} \boldsymbol{v_1} \\ \vdots \\ \boldsymbol{v_n} \end{bmatrix} + \begin{bmatrix} \boldsymbol{w_1} \\ \vdots \\ \boldsymbol{w_n} \end{bmatrix}.$$

One will notice that this an invertible transformation.

• <u>Def:</u> A **dilation** or **scaling** is a map:

$$\boldsymbol{T : V \to V}$$

$$\boldsymbol{v \mapsto \lambda v}$$

for some scalar $\boldsymbol{\lambda} \in \boldsymbol{F} := \mathbb{R}$ (the underlying field). This is another invertible transformation (since we are dealing with a *field*). In coordinates this is just:

$$\begin{bmatrix} \boldsymbol{v_1} \\ \vdots \\ \boldsymbol{v_n} \end{bmatrix} \mapsto \boldsymbol{\lambda} \begin{bmatrix} \boldsymbol{v_1} \\ \vdots \\ \boldsymbol{v_n} \end{bmatrix}.$$

In more general context, when the terms are used simultaneously, **dilation scaling** refers to:

$$\boldsymbol{u(x,t) \mapsto \lambda^{\alpha} u(\lambda^{\beta} x, \lambda^{\gamma} t)}.$$

• <u>Def:</u> A **reflection in the $\boldsymbol{i^{th}}$-slot** is a map:

$$\boldsymbol{T : V \to V}$$

$$\boldsymbol{v \mapsto Tv}$$

where in the basis, we have the invertible (idempotent) matrix expression:

$$[\boldsymbol{Tv}]_{\beta} = \begin{bmatrix} \boldsymbol{1} & \boldsymbol{0} & \boldsymbol{\dots} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \ddots & & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\dots} & \boldsymbol{-1} & \boldsymbol{\dots} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\dots} & \boldsymbol{0} & \boldsymbol{1} \end{bmatrix} \begin{bmatrix} \boldsymbol{v_1} \\ \vdots \\ \boldsymbol{v_i} \\ \vdots \\ \boldsymbol{v_n} \end{bmatrix}.$$

• Def: A **θ-radian, CCW-rotation about the i^{th}-axis** is a map:

$$T : V \to V$$

$$v \mapsto Tv$$

whose coordinate expression in general is hard to get at. In 3 dimensions we have for example rotation about x, y, and z:

$$[T]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}, \quad = \begin{bmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix}, \quad = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These are invertible maps [Exercise: Prove this! The inverse matrices just have $-\theta$ inserted.] Intuitively, rotations have start and stop vectors as well as an invariant axis. So the rotations happen in a 2D subspace. This manifests as the sine and cosine being in different locations with the fixed axis having a 1.

In lieu of a good general coordinate expression, to work with the higher dimensional cases, we try to encompass rotation operators with the following:

• Def: An **orthogonal transformation** is one whose matrix representation has all mutually orthonormal columns (w.r.t. the standard inner product). To put this in symbols:

$$\langle [T]_{\beta,i} \ , \ [T]_{\beta,j} \rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

These have other characterizations as well, such as $A^t A = I_n$, where $A := [T]_\beta$ but the point is we abstract a property of the above matrices.

Translations, dilations, reflections, and rotations may be composed to get more complicated ones. We don't concern ourselves with this here. Next we look at some other transformations provided by...

II.2.2: Representations of the Symmetric Group

• Def: The **Symmetric Group** is the collection of all **permutations on n-indices**, which in fact forms the algebraic structure of a *group*. They can be uniquely displayed with **increasing cycle notation**, such as:

$$\mathbf{(2\ 5\ 6),\ (1\ 2\ 3\ 4),\ (1),\ (2\ 7),}\ \text{etc.}$$

These read from left to right as such: *"2 goes to 5, 5 goes to 6, 6 goes to 2"* etc. When we deal with algebraic operations between them such as their composition or formal linear combinations over a field, it helps to save the information into variables (conventionally greek symbols):

$$\boldsymbol{\sigma := (2\ 5\ 6),\ \tau := (1\ 2\ 3\ 4),\ etc.}$$

• Def: A **group representation** is a **group homomorphism**:

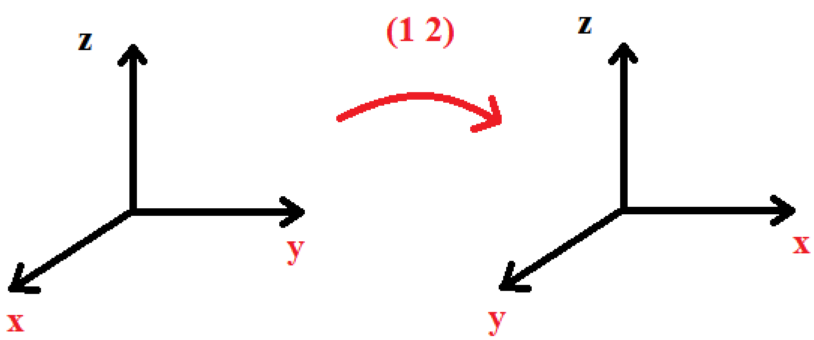
$$\boldsymbol{\rho : G \to GL_n(V)}$$

where $\boldsymbol{GL_n(V)}$ is the group of invertible linear operators on a vector space of our choice. Of course, when we choose a basis, we are mapping the group elements into their corresponding invertible matrices.

• Def: In one case for $\boldsymbol{G := S_n}$, these matrices, $[\boldsymbol{\rho(\sigma)}]_\beta$ are called **permutation matrices**.

- -

Letting $\boldsymbol{V} := \mathbb{R}^{\boldsymbol{n}}$, we can define "a" representation by acting the permutations on the axes of the space. For example in the figure we portray a 2-cycle in 3-space:



If we collect the images of the bases vectors (taken to be axes), we get the associated matrix:

$$[\boldsymbol{\rho}(\mathbf{12})]_{\boldsymbol{\beta_{std.}}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}.$$

Similarly, we obtain all the other 2-cycles. And conveniently, these generate all of $\boldsymbol{S_n}$ so we have the full map detailed by matrix multiplication (appeal to group hom). Note that these are **orthogonal matrices** and that products of orthogonal matrices are orthogonal (easy proof). There are other representations of $\boldsymbol{S_n}$ of course. See a Rep. Theory text for more. Let's take a step back next page...

II.2.3: Method of Undetermined Coefficients

Method: Guess a particular form of solution. Plug it into the PDE and see what the resulting equality tells you about your coefficients. Apply auxiliary data later to solve for unique coefficients etc. Some examples using this method are listed below:

• Rotation Invariance and Dilation Scaling Invariance:

In the Laplace and Heat equations, we were able to assume the solution was of the particular form:

$$\boldsymbol{u := u(r) = u(|x|_2)} \qquad \text{resp.} \qquad \boldsymbol{u := t^{-\alpha}u(|t^{-\beta}x|_2)},$$

by naively appealing to the observations that the corresponding PDE's held rotational invariance (resp. dilation scaling invariance) among their coordinate expressions. Upon substitution of these special solution forms, the PDE's reduced to ODE's and then were solvable by lesser means. A similar appeal to dilation scaling is done in a solution to the Porous Medium Equation (p.185 [5]).

• Traveling Waves and Plane Waves:

• Def: (p.176 [5]) For a PDE with real variables $\boldsymbol{(x_1, ..., x_n, x_{n+1} = t)}$, a solution of the form:

$$\boldsymbol{u(x,t) = v(y \cdot x - \sigma t)},$$

where $\boldsymbol{y} \in \mathbb{R}^{\boldsymbol{n}}$ and $\boldsymbol{\sigma} \in \mathbb{R}$ is called a plane wave. Additionally we say the:
> **Wavefront** is normal to $\boldsymbol{y}$,
> **Wave Numbers** are the $\boldsymbol{\{y_i\}_{i\in\{1,\dots,n\}}}$,
> **Time Frequency** is $\boldsymbol{\sigma}$,
> **Speed** is given by $\frac{\boldsymbol{\sigma}}{|\boldsymbol{y}|}$, and the
> **Profile** is given by the function $\boldsymbol{v}$.

In the special case where $n = 1$, this solution form is called a **traveling wave**.

We will see many examples of complex-valued, **exponential plane wave solutions**,

$$\boldsymbol{u(x,t) = e^{i(y\cdot x - \sigma t)}},$$

with $\boldsymbol{\sigma} \in \mathbb{C}$, used for solving Linear PDE's with M.o.U.C.! To name a few: Heat, Wave, Klein-Gordon, Schrodinger, and Airy can be solved this way. Other traveling wave solutions can be observed as well in KdV and Scalar Reaction-Diffusion (Section 4.2 [5]).

[Research Problem: Study the relationship between the solution forms we've encountered and the respective coordinate transformations developed in this Section (II.2). Consider as well, applying permutations to multi-variable polynomials.]

Breathe!

III: Technical Results Entries

The purpose of this section is to be a toolbox that is distinct from the more theoretical toolbox of Section II. When something is too lengthy to put in the discussion without ruining the flow, it goes here.

Sub-Table of Contents:

Entry 1: Statement of the Chain Rule

• **Prop:** Let $\boldsymbol{f} : \mathbb{R}^{\boldsymbol{m}} \to \mathbb{R}^{\boldsymbol{n}}$ and $\boldsymbol{g} : \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}$ be continuous mappings of Euclidean spaces such that their composition:

$$\boldsymbol{g} \circ \boldsymbol{f} : \mathbb{R}^{\boldsymbol{m}} \to \mathbb{R}$$

is well-defined. Then the following formula holds:

$$\partial_i(g \circ f) = \sum_{j=1}^{n} \partial_j g(f^1, \ldots, f^n) \cdot \partial_i f^j.$$

Weak Proof: Atomically, if $g := f^1 \cdot f^2$, the formula yields:

$$\partial_i(g \circ f) = f^2 \cdot \partial_i f^1 + f^1 \cdot \partial_i f^2,$$

which agrees with the known *product rule* from basic calc. [Exercise: The real proof should be a difference quotient argument.] □

We will use this formula countless times!

Entry 2: Differentiating Integrals

This results entry was influenced by a discussion of the Reality Check (from the Non-Hom. Transport solution), that I asked on Math Stack Exchange [13]. They quoted ***Liebniz' Integration Rule****, which led me to reference [12], but I didn't believe what I found initially, so I went back to first principles, namely the definition of derivative.*

• Prop: For a continuous function of two variables, $f(s,t)$, the time derivative of the s-integral with t as the upper limit is given by:

$$\partial_t \int_0^t f(s,t)ds = f(t,t) + \int_0^t \partial_t f(s,t)ds.$$

Proof: Writing out the difference quotient:

$$\partial_t \int_0^t f(s,t)ds := \lim_{h\to 0} \frac{1}{h}\Big[\int_0^{t+h} f(s,t+h)ds - \int_0^t f(s,t)ds\Big].$$

Adding by zero:

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_0^{t+h} f(s,t+h)ds - \int_0^t f(s,t)ds + \int_t^{t+h} f(s,t)ds - \int_t^{t+h} f(s,t)ds\right].$$

Combining the second and fourth terms:

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_0^{t+h} f(s,t+h)ds - \int_0^{t+h} f(s,t)ds + \int_t^{t+h} f(s,t)ds\right].$$

Combining the first two and distributing the limit and the scalar yields:

$$= \lim_{h\to 0} \int_0^{t+h} \frac{1}{h}\big[f(s,t+h) - f(s,t)\big]ds + \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f(s,t)ds$$

$$= \int_0^t \partial_t f(s,t)ds + \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f(s,t)ds\ .$$

- -

Now the tricky part is handling this mysterious indeterminate form on the right. We split the limits with a point $s_0 \in (t, t+h)$ and separate into two integrals, applying $\left(\int_a^b = -\int_b^a\right)$:

$$\int_t^{t+h} f(s,t)ds = \int_{s_0}^{t+h} f(s,t)ds - \int_{s_0}^{t} f(s,t)ds \qquad (\star)$$

Define $F(u,t) := \int_{s_0}^u f(s,t)ds$. Then $(\star) = F(t+h,t) - F(t,t)$. Applying the limit and scalar gives:

$$\lim_{h\to 0}\frac{1}{h}[F(t+h,t) - F(t,t)] =: \partial_u F(u,t)\big|_{u=t} = f(t,t)$$

by the **Fundamental Theorem of Calculus Part 1**, (pg.381 [7]). ■ (Discussion Continues)

Notice that in the above, we are free to assume $\boldsymbol{f}$ has any number of extra variables (as long as they are independent of $\boldsymbol{s}$ and $\boldsymbol{t}$. So the result applies to $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{t})$ as well, where $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$ (as in the motivating problem). The extra variables literally just get carried through the entire proof.

More generally, one can assume the upper limit is a function of $\boldsymbol{t}$ instead (not just a basic linear function), following through with the proof, you will notice the chain rule comes into play and we get a similar result. [Exercise: Carry this out!]

Entry 3: (Q&A) A System of PDEs in Stoke's Theorem

The context for this discussion is a Q and A that I answered on Stack [14].

Question:

I am doing some exercises and I don't understand what is wrong with my solution here.

The problem is: given the integral

$$I = \int_S (1+x^2)f(x)dydz - 2xyf(x)dzdx - 3zdxdy$$

Find such a continuously differentiable function f such that the integral I is equal for all surfaces S, whose border is a circle $C := \{(\cos t, \sin t, 1) \mid t \in [0, 2\pi]\}$ and then calculate the integral I.

My thinking is that any function that is defined everywhere and C^1 should be alright! Let's denote $\overrightarrow{R}$ the vector field over which we are integrating. For any well defined C^1 function $f(x)$, field $\overrightarrow{R}$ will be well defined and C^1. Then by Stokes theorem we have:

$$\int_S rot\overrightarrow{R}d\vec{S} = \int_{\partial S} \overrightarrow{R}d\vec{r}$$

where $\partial S = C$, which is a fixed number, so the integral on the left will be equal for every surface S with the same border.

However, the solution uses Gauss's theorem instead and shows that the sufficient condition is that $div\overrightarrow{R} = 0$. It also states (without proof) that this condition is also necessary.

To sum up, I would appreciate if you help me figure out

1) What is wrong with my reasoning using Stokes theorem? 2) How to show that $div\overrightarrow{R} = 0$ is a necessary condition?

My Initial Comment:

Hypotheses aside, the general version of this theorem looks like: $\int_X d\omega = \int_{\partial X} \omega$. So the answer to this question requires finding ω with the given info. I.e. design ω with choice of $f(x)$ such that $d\omega$ is as above. What you have written for the integral equality is missing "curl" and dot products. This is not divergence theorem. If we try to apply the Poincare Lemma, we need to show $d(\text{“}d\omega\text{”}) = 0$, this forces $f(x) = 3tan^{-1}(x)$ for it to be closed. Though this doesn't give the ω we need for the rest of the problem.

(Continues)

My Answer:

The expression in a chart for a differential 1-form (using Einstein conv.) looks like:

$$\omega = \omega_i dx^i$$

$$\implies d\omega = d\omega_i \wedge dx^i = \partial_j \omega_i dx^j \wedge dx^i.$$

For $n = 3$, we have (invoking multilinearity and skew-symmetry of $\wedge$) that:

$$d\omega = (\partial_x \omega_i dx \wedge dx^i) + (\partial_y \omega_i dy \wedge dx^i) + (\partial_z \omega_i dz \wedge dx^i)$$

$$= \Big(\partial_x \omega_1 dx \wedge dx + \partial_x \omega_2 dx \wedge dy + \partial_x \omega_3 dx \wedge dz\Big)$$
$$+\Big(\partial_y \omega_1 dy \wedge dx + \partial_y \omega_2 dy \wedge dy + \partial_y \omega_3 dy \wedge dz\Big)$$
$$+\Big(\partial_z \omega_1 dz \wedge dx + \partial_z \omega_2 dz \wedge dy + \partial_z \omega_3 dz \wedge dz\Big)$$

$$= (\partial_x \omega_2 - \partial_y \omega_1) dx \wedge dy + (\partial_y \omega_3 - \partial_z \omega_2) dy \wedge dz + (\partial_x \omega_3 - \partial_z \omega_1) dx \wedge dz$$

$$= (\partial_y \omega_3 - \partial_z \omega_2) dydz + (\partial_z \omega_1 - \partial_x \omega_3) dzdx + (\partial_x \omega_2 - \partial_y \omega_1) dxdy.$$

Since we have:

$$d\omega := (1 + x^2) f(x) dydz - 2xyf(x) dzdx - 3z dxdy$$

Combining with the above gives a system that ω satisfies:

$$(i)\ \partial_y \omega_3 - \partial_z \omega_2 = (1 + x^2) f(x)$$

$$(ii)\ \partial_z \omega_1 - \partial_x \omega_3 = -2xyf(x)$$

$$(iii)\ \partial_x \omega_2 - \partial_y \omega_1 = -3z$$

(Continues)

Ans (Continued):

Now, bear with me as I proceed to find (a) solution to this system (not claiming uniqueness). If we assume that:

$$\omega_3 \equiv 0,$$

then (i) and (ii) simplify to:

$$\partial_z \omega_2 = -(1+x^2)f(x)$$
$$\partial_z \omega_1 = -2xyf(x)$$

Integrating these with respect to z gives:

$$\omega_1 = -2xyzf(x) + G_1(x,y)$$

$$\omega_2 = -z(1+x^2)f(x) + G_2(x,y)$$

where $G_i(x,y)$ are constants of integration w.r.t. z.

Let's assume these vanish identically as well. Then combining these results with (iii) gives:

$$\partial_x \omega_2 - \partial_y \omega_1 = -z[2xf(x) + (1+x^2)f'(x)] + 2xzf(x)$$

$$= -z(1+x^2)f'(x) = -3z$$

Assuming $z \neq 0$ and $x \neq i$ gives:

$$f'(x) = \frac{3}{1+x^2}$$

Or equivalently:

$$f(x) = 3tan^{-1}(x) + C.$$

This is a good reaffirmation of the exactness discussion! Take $C = 0$.

Putting this all together gives (a) primitive:

$$\omega = (-6xyz \cdot tan^{-1}(x))dx + (-3z(1+x^2) \cdot tan^{-1}(x))dy$$

Provided $z \neq 0$ (which won't matter over C).

Finally, in your application of Stoke's Theorem:

$$I = \int_S d\omega = \int_{\partial S = C} \omega$$

So dump in the parameterization for C $\Big($i.e. $(x,y,z)(t) = (cos(t), sin(t), 1)\Big)$ for $t \in [0, 2\pi]$ and the result will follow! ■

Entry 4: (Q&A) On Calculating M.o.C.-System for Conservation Laws

The context for this discussion is p.113 of Evan's PDE text and my Q and A that I self answered on Stack [15].

Question:

In general, for the *(Nonlinear, First Order) Method of Characteristics*, we have a parameterized curve:

$$x(s) = (x^1(s), ..., x^n(s))$$

and some related variables , for an unknown (smooth enough) function, $u : \mathbb{R}^n \to \mathbb{R}$:

$$z(s) := u(x(s))$$

$$p^i(s) := \partial_{x^i} u(x(s)),$$

for $i \in \{1, ..., n\}$.

With these definitions, one proceeds to a certain system of ODE's that I won't list here.

In a special case relating to the **Hamilton-Jacobi** equations, we have **time** as a separate domain variable, which changes the above to:

$$y(s) := (x(s), t(s)) = \big(y^1(s), ..., y^n(s), y^{n+1}(s)\big)$$

$$z(s) := u(y(s))$$

$$q(s) := \big(q^1(s), ..., q^n(s), q^{n+1}(s)\big)$$

where

$\forall i \leq n,\ y^i(s) := x^i(s)$;
$y^{n+1}(s) := t(s)$;
$\forall i \leq n,\ q^i(s) := p^i(s)$; and
$q^{n+1}(s) := \partial_{n+1} u(x(s))$.

This time the unknown function is $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$.

(Continues)

Q (Continued):

Consider now the PDE, $\boldsymbol{F}((\boldsymbol{x},\boldsymbol{t}),\boldsymbol{u},\boldsymbol{Du}) = \boldsymbol{0}$, instantiated as:

$$\boldsymbol{u}_t + \boldsymbol{H}(\boldsymbol{x},\boldsymbol{Du}) = \boldsymbol{0}.$$

If we restrict to the curve, $\boldsymbol{x}(\boldsymbol{s})$, this looks like $\boldsymbol{F}(\boldsymbol{y},\boldsymbol{z},\boldsymbol{q}) = \boldsymbol{0}$:

$$\left[\ \boldsymbol{q}^{n+1}(\boldsymbol{s}) + \boldsymbol{H}(\boldsymbol{x}(\boldsymbol{s}),\boldsymbol{p}(\boldsymbol{s})) = \boldsymbol{0}\ \right],$$

with $\boldsymbol{x}$ and $\boldsymbol{p}$ understood as the first $\boldsymbol{n}$ coordinates of $\boldsymbol{y}$ and $\boldsymbol{q}$ respectively.

> > When computing the RHS for the system of ODE's (not listed), the author states: $\boldsymbol{\partial_z F} = \boldsymbol{0}$. At first glance, you don't see $\boldsymbol{z}$ in the equation, but recall we defined $\boldsymbol{q^i}(\boldsymbol{s}) = \boldsymbol{\partial_{y^i} z}(\boldsymbol{s})$. So there are contributions from both terms $\left(\boldsymbol{q}^{n+1},\boldsymbol{H}(\boldsymbol{x},\boldsymbol{p})\right)$ that require [computation of]:

$$\boldsymbol{\partial_z(\partial_{y^i} z)}.$$

How does one further evaluate?

My Answer:
If we write out the difference quotient for z (using function variables), we get:

$$\boldsymbol{\partial_z(\partial_{y^i} z(s))} := \lim_{h(s)\to 0(s)} \frac{(\partial_{y^i})(z(s)+h(s)) - (\partial_{y^i})(z(s))}{h(s)}$$

$$= \lim_{h(s)\to 0(s)} \frac{(\partial_{y^i})(z(s)) + (\partial_{y^i})(h(s)) - (\partial_{y^i})(z(s))}{h(s)}$$

$$\lim_{h(s)\to 0(s)} \frac{(\partial_{y^i})(h(s))}{h(s)} \qquad (\star)$$

by Linearity of the partial operators, $\boldsymbol{\partial_{y^i}(\cdot)}$, in their function variable. But

$$\boldsymbol{\partial_{y^i} h(s) = 0(s)}$$

since $\boldsymbol{h(s)}$ doesn't contain $\boldsymbol{y^i}$ as a variable (technicality here). So the desired limit $(\star)$ is zero.

This gives:

$$\boldsymbol{\partial_z F(y,z,q) = \partial_z q^{n+1} + \partial_z H(x,p) = \partial_z(\partial_{y^{n+1}} z) + \sum_{j=1}^{n} \partial_{p^j} H(x,p)\cdot\partial_z(\partial_{y^j} z) = 0.}\ \blacksquare$$

(Back to Solutions Catalog)

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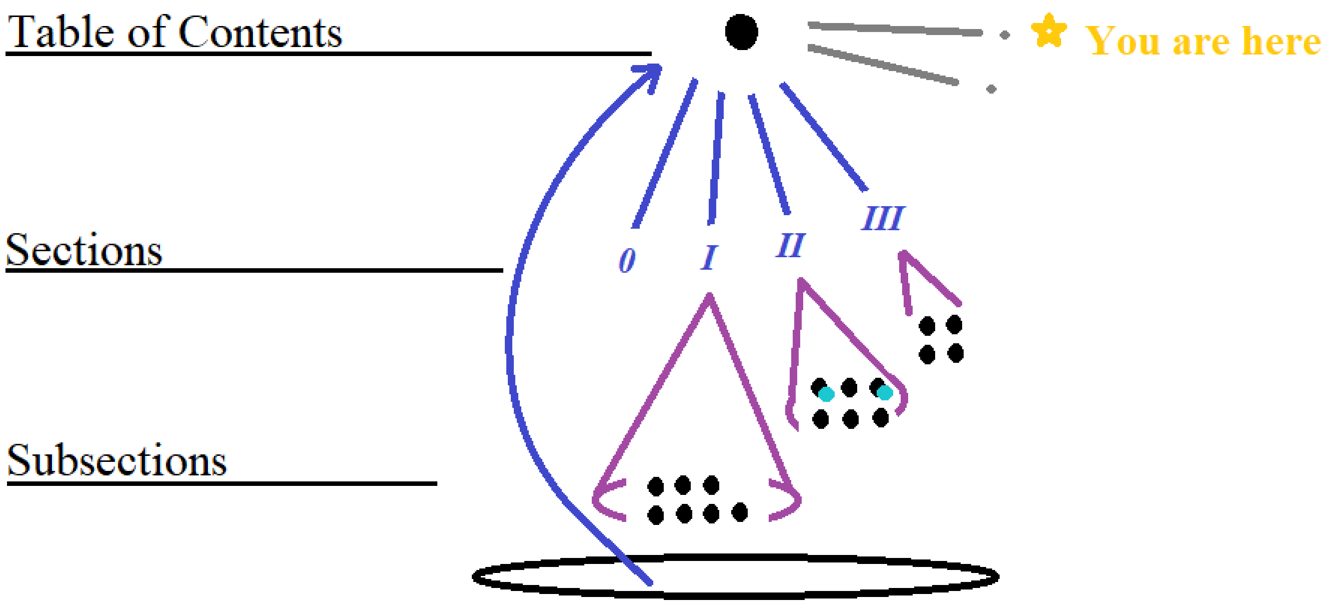

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Oh, and preliminary knowledge required to read this is say Calculus III and Real Analysis (see my other release for R.A. and Topology)!