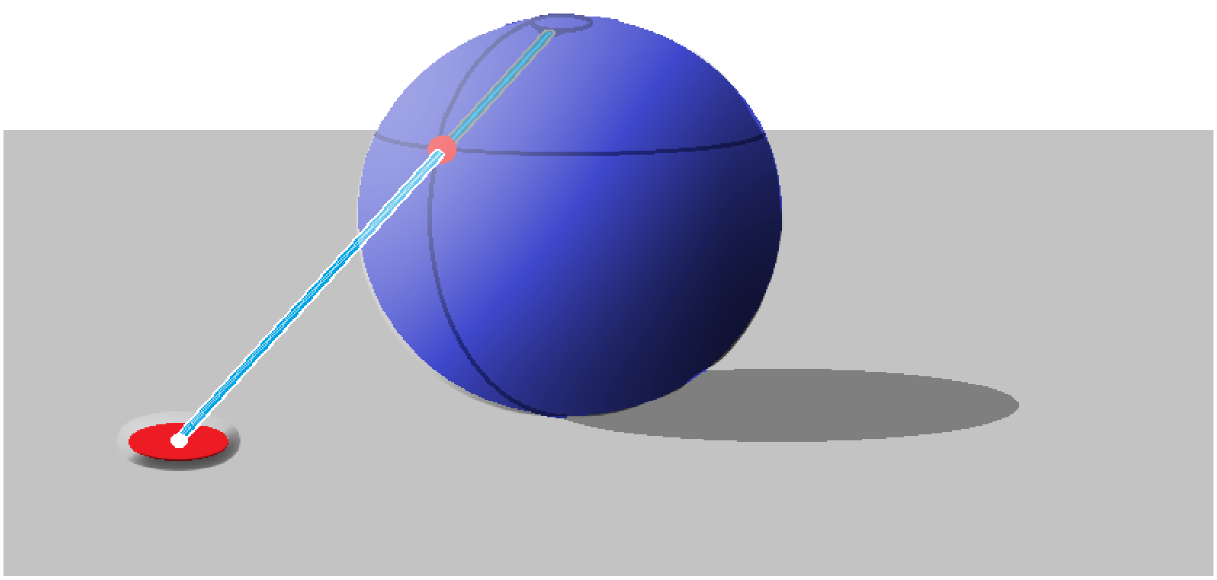
COMPLEX ANALYSIS

By: Kevin Smith

1 • Limits and Complex Differentiability
2 • Zeros, Singularities, and Series Expansions
3 • Paths, Winding Numbers, and Contour Integrals
4 • Practical Results



1. Limits and Complex Differentiability

• Def: A complex series $\{z_n\}_{n=0}^{\infty}$ is **convergent** to $z \in \mathbb{C}$ if $\forall \varepsilon > 0\ \exists N$ such that $\forall n \geq N$, $|z - z_n| < \varepsilon$.

• Def: A series $\{f_n(z)\}_{n=0}^{\infty}$ is **uniformly convergent** to $f(z)$ if $\forall z \in \mathbb{C}, \varepsilon > 0\ \exists N$ such that $\forall n \geq N$, $|f(z) - f_n(z)| < \varepsilon$.

• Def: The **limit of $f(z)$** as z approaches z_0 exists and equals L, denoted $\lim\limits_{z \to z_0} f(z) = L$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall w \in \mathbb{C}$ we have $|z - w| < \delta \implies |f(z) - f(w)| < \epsilon$.
Notice then that if the limit exists at z_0 and $f(z_0) = L$, then f is **continuous at z_0**.

• Def: $\lim\limits_{z \to \infty} f(z) := \lim\limits_{z \to 0} f(1/z)$.

As well, we have: $\lim\limits_{z \to a} f(z) = \infty$ if $\forall \varepsilon\ \exists \delta,\ \forall z \in B(a;\delta) \implies f(z) \in \mathbb{C} - B(f(a);\varepsilon)$.

• Def: We define the **derivative of $f(z)$ at z_0** by:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z)$$

If such a limit exists, we say f is **complex differentiable**, **holomorphic**, or **analytic** at the given point (synonyms). This can be generalized to verbage on *open regions* $G \subseteq \mathbb{C}$ with special case $G = \mathbb{C}$ having special name for f being **entire**.

• Def: If $G \subseteq \mathbb{C}$ is an open region and f is a function defined and analytic in G except at *poles* (to be defined later), then f is called **meromorphic** on G.

• Thm/Def: The following **Cauchy-Riemann Equations** must be satisfied by analytic functions:

Writing $f(x + iy) = u(x, y) + iv(x, y)$, then
$u_x = v_y$ and $u_y = -v_x$

We say for $f \in C^2(\mathbb{R})$, that f is **harmonic** if it satisfies $\Delta f = f_{xx} + f_{yy} = 0$.
Taking second derivatives of the C.R. equations and adding, we see that both component functions u and v are harmonic. In the event that two harmonic functions u and v exist making $f = u + iv$ analytic, we say u and v are **harmonic conjugates**.

2. Zeros, Singularities, and Series Expansions

• <u>Def:</u> If $f : G \to \mathbb{C}$ is analytic and $a \in G$ satisfies $f(a) = 0$ then a is a **zero of multiplicity** $m \geq 1$ if there is an analytic function $g : G \to \mathbb{C}$ such that $\boldsymbol{f(z) = (z-a)^m g(z)}$ where $g(z) \neq 0$.

• <u>Def:</u> A function f has an **isolated singularity** at $z = a$ if $\exists R > 0$ such that f is defined and analytic on $B(a; R) - \{a\}$, but not at a. The point a is called a **removable singularity** if there is an analytic function $g : B(a; R) \to \mathbb{C}$ such that g(z)=f(z) for $\forall z \in B(a; R) - \{a\}$.

★ <u>Theorem 1.2 (pg. 103 Conway):</u> If f has an isolated singularity at a then a is **removable** iff:

$$\boldsymbol{\lim_{z \to a} (z-a) f(z) = 0}$$

• <u>Def:</u> If $z = a$ is an isolated singularity of f then a is a **pole** of f if $\boldsymbol{\lim_{z \to a} |f(z)| = \infty}$.

If f has a pole at $z = a$ and m is the smallest positive integer such that $f(z)(z-a)^m$ has a removable singularity at $z = a$, then f has a **pole of order** $\boldsymbol{m}$ at $z = a$.

• <u>Def:</u> If an isolated singularity is neither a pole nor a removable singularity, it is called an **essential singularity**.

• <u>**Laurent Series Development**</u> (See pg.107 for computing coefficients in terms of integrals)

More practically, we utilize the power series $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ convergent for $|z| < 1$ to compute Laurent series for given functions by transforming our expressions into ones of the corresponding form above. More often then not, partial fraction decomposition is also utilized.

<u>Examples:</u>

1.) Say we have a function of the form:

$$\boldsymbol{\frac{1}{a-b(z-c)}} = \frac{1}{a} \cdot \frac{1}{1-\left(\frac{b}{a}(z-c)\right)} = \boldsymbol{\frac{1}{a} \sum_{n=0}^{\infty} \left(\frac{b}{a}(z-c)\right)^n}$$

This is convergent on $|\frac{b}{a}(z-c)| < 1$ or in other words **convergent on** $\boldsymbol{B(c; |\frac{a}{b}|)}$.

2.) Similarly, if we have:

$$\boldsymbol{\frac{1}{a-b(z-c)}} = \frac{-1}{b(z-c)} \cdot \frac{1}{1-\frac{a}{b(z-c)}} = \boldsymbol{\frac{-1}{b(z-c)} \sum_{n=0}^{\infty} \left(\frac{a}{b} \cdot \frac{1}{(z-c)}\right)^n}$$

This is convergent on $|\frac{a}{b} \cdot \frac{1}{(z-c)}| < 1$ or in other words **convergent on** $\mathbb{C} - \overline{\boldsymbol{B(c; |\frac{a}{b}|)}}$.

This second sum can be rewritten with negative indices so that we get a more complete series look of the form $\sum_{-\infty}^{\infty} a_n (z-c)^n$. This reduces the problem of finding the Taylor and Laurent series to applying the appropriate formulas above to functions analytic in the corresponding regions of convergence.

• <u>Def:</u> The **residue of** $\boldsymbol{f}$ **at z=c**, denoted by $\boldsymbol{Res(f, c)}$ is just the -1 power term in the Laurent expansion.

3. Paths, Winding Numbers, and Contour Integrals

• Def: A **path** from a to b in $\mathbb{C}$ is just a map $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

We usually require *regularity class* C^0 that way piecewise integration is possible (i.e. functions pre-composed with C^0 remain integrable). There is mention of more general *rectifiable paths*, we won't go there. Also, sometimes *smooth* is used to mean continuous (not C^∞, watch out!). For what follows, when you see "rectifiable" just think γ is piecewise continuous.

• Def: A **curve** is the geometric abstraction of a path in the sense that it is an equivalence class of paths up to *re-parameterization*. However, the distinction in terminology is blurred.

• Def: The **trace** of a path $\gamma : [0,1] \to \mathbb{C}$ is defined as $\boldsymbol{\{\gamma\} := \gamma([0,1])}$.

• Def: If γ is a closed rectifiable curve in $\mathbb{C}$ then $\forall a \notin \{\gamma\}$,

$$\boldsymbol{n(\gamma; a) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz}$$

is called the **index of γ with respect to a**. It is also called the **winding number of** γ **around** a (counts the number of times γ wraps around a in the clockwise direction).

For simple closed curves, $n(\gamma; a)$ can be found visually. These will be useful for calculations of integrals with Cauchy's Formulas (see results section for formulas).

• Def: For a smooth curve γ and a complex valued function $f \in C^0(\{\gamma\})$, we define the **line integral** as:

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt$$

This is also called a **contour integral**.

There are initially two ways to calculate such beasts, either use the fundamental theorem for line integrals *(see Thm IV.1.18) when the* primitive *is known or dump in the parameterization for γ and seperate real and imaginary parts of the integral making it two real integrals.*

Otherwise, we have to explore other techniques using Cauchy's formulas and winding numbers, partial fraction decomposition, series expansions, and/or other real analysis tricks once we reduce to that case.

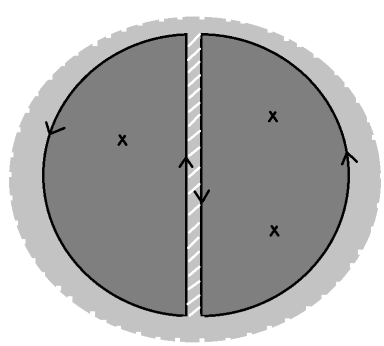


Figure: Schematic for avoiding singularities in contour integrals.

4. Results (Compressed/Expanded)

[1] • $f' \equiv 0$ and analytic $\implies$ constant (III.2.9: p.37)
[2] • $Exp(z)$ and $Log(z)$, branches (See: III: p.39)
[3] • Conformal Maps (III. See p.46)
[4] • Mobius Transformations (III. See p.47)

[5] • Prop IV.1.17 (IV: p.65)
[6] • Theorem (IV.1.18: p.65 FTCforLI)
[7] • Theorem IV.2.8 (p.72) Analyticity and Series Expansion
[8] • Cauchy's Estimate (IV.2.14: p.73)

[9] • Liouville's Theorem (IV.3.4: p.77)
[10] • Zeros Theorem (IV.3.7: p.78)
[11] • Cauchy's Integral Formula's (IV.5.4/6/((8))) p.84-86)
[12] • Morera's Theorem (IV.5.10 p.86)

[13] • Classifying Isolated Singularities Theorem (V.1.18: p.109)
[14] • Residue Theorem (V.2.2: p.112)
[15] • Handling Infinite Limits in Integrals (V: See p.113)
[16] • Rouche's Theorem (V: p.125)

[17] • Argument Principle (V: p.123)
[18] • Max Mod Theorem (VI.1.1: p.128)

4. Results (Compressed/Expanded)

• Prop III.2.10 (p.37):
If G is open and connected and $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $z \in G$, then f is constant.

• On Exp(z) and Log(z) (III: p.39):

We effectively define:

$f(z) = \boldsymbol{Exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}}$ and

$g(z) = \boldsymbol{Log(z) := ln(|z|) + iArg(z)}$, where $Arg(z) = \left(tan^{-1}\left(\frac{Im(z)}{Re(z)}\right)\right) \; (mod \pi)$

The above log definition is known as the **principle branch**. Other branches are given by:

• Def: If G is an open connected set in $\mathbb{C}$ and $f : G \to \mathbb{C}$ is a continuous function such that $z = e^{f(z)}$ for all $z \in G$, then f is a **branch of the logarithm**.

One will notice that we get a class of branches of log by translating $Arg(z)$ by $2\pi k$ for $k \in \mathbb{Z}$. Other classes are given by precomposing with a homeomorphism.

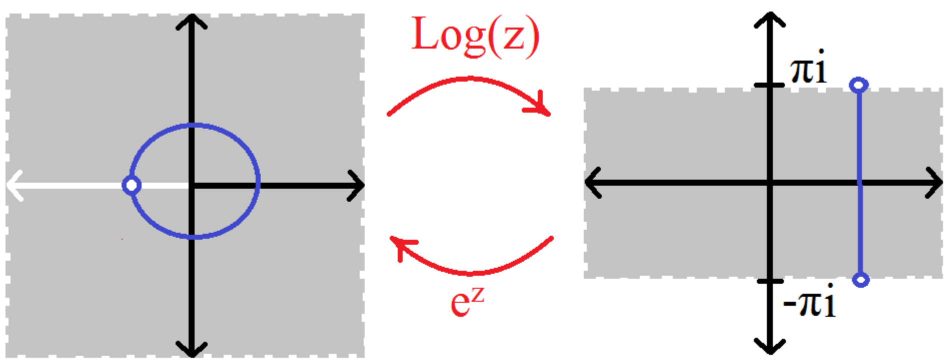


Figure: Domains of Log and Exp
With their actions on lines and circles.

• On Conformal Maps (Thm III.3.4: p.46):
If $f : G \to \mathbb{C}$ is analytic then f preserves angles at each point z_0 of G where $f'(z_0) \neq 0$.
If f is analyitic and $f'(z) \neq 0$ for all $z \in G$ then f is **conformal**. The converse is also true.

• Mobius Transformations (III. See p.47):
Def: A mapping of the form $\boldsymbol{S(z) = \frac{az+b}{cz+d}}$ is called a **linear fractional transformation**. If $\{a, b, c, d\} \in \mathbb{C}$ also satisfy $ad - bc \neq 0$ then $S(z)$ is called a **Mobius Transformation**. The basic ones are given below. Note that they close under composition.

$S(z) = z + a$ is a **translation**,
$S(z) = az$ (with $a > 0$) is a **dilation**,
$S(z) = e^{i\theta} z$ is a **rotation**, and
$S(z) = \frac{1}{z}$ is **inversion**.

• Prop IV.1.17 (IV: p.65): (Special Case)
Let γ be a smooth curve and suppose that f is a function that is continuous on $\{\gamma\}$.
Then: $|\int_\gamma \boldsymbol{f}| \leq \boldsymbol{max\{|f(z)| : z \in \{\gamma\}\} \cdot length(\gamma)}$

• Theorem (1.18: p.65 FTCforLI):
Let G be open in $\mathbb{C}$ and let γ be rectifiable in G with initial and terminal points α and β respectively. If $f : G \to \mathbb{C}$ is a continuous function with a primitive $F : G \to \mathbb{C}$, then:

$$\int_\gamma \boldsymbol{f = F(\beta) - F(\alpha)}$$

•Theorem IV.2.8 (p.72) Analyticity and Series Expansion:
Let f be analytic in $B(a; R)$, then $\boldsymbol{f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n}$ for $z \in B(a; R)$, where $\boldsymbol{a_n = \frac{1}{n!} f^{(n)}(a)}$ and this series has radius of covergence $\geq R$.

• Cauchy's Estimate (Cor. IV.2.14: p.73):
For f analytic in $B(a; R)$ with $\forall z \in B(a; R)$, $|f(z)| \leq M$, we have:

$$\boldsymbol{|f^{(n)}(a)| \leq \frac{n! \cdot M}{R^n}}$$

•Liouville's Theorem (IV.3.4: p.77):
If f is a bounded entire function, then f is constant.

• Zeros Theorem (IV.3.7: p.78):
Let G be a connected open set and let $f : G \to \mathbb{C}$ be an analytic function. Then TFAE:

(a) $f \equiv 0$,
(b) $\exists a \in G$ with $\forall n \geq 0, f^(n)(a) = 0$, and
(c) $\{z \in G \mid f(z) = 0\}$ has a limit point in G.

• Cauchy's Integral Formula's (IV.5.8) p.84-86):
Let G be an open subset of the plane, $f : G \to \mathbb{C}$ analytic. If $\gamma_1, \ldots, \gamma_m$ are closed rectifiable curves in G such that $\sum_{k=1}^{m} n(\gamma_k; w) = 0 \; \forall w \in \mathbb{C} - G$, then we have $\forall a \in G - \{\gamma\}$ and $\forall k \geq 1$:

$$\boldsymbol{\frac{2\pi i}{k!} f^{(k)}(a) \cdot \sum_{j=1}^{m} n(\gamma_j; a) = \sum_{j=1}^{m} \int_{\gamma_j} \frac{f(z)}{(z-a)^{k+1}} dz}$$

• Morera's Theorem (IV.5.10 p.86):
Let G be a region and let $f : G \to \mathbb{C}$ be a continuous function such that for every triangular path $T \subseteq G$, $\int_T f = 0$, then f is analytic in G.

• Classifying Isolated Singularities Theorem (V.1.18: p.109):
Let $z = a$ be an isolated singularity of f and let $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$ be its Laurent Expansion in $ann(a; 0, R)$. Then:

(a) $z = a$ is a **removable singularity** iff $a_n = 0, \forall n \leq -1$ (i.e. no negative power terms),
(b) $z = a$ is a **pole of order m** iff $a_{-m} \neq 0$ and $a_n = 0, \forall n \leq -(m+1)$ (i.e. m negative terms), and
(c)$z = a$ is an **essential singularity** iff $a_n \neq 0$ for infinitely many negative integers.

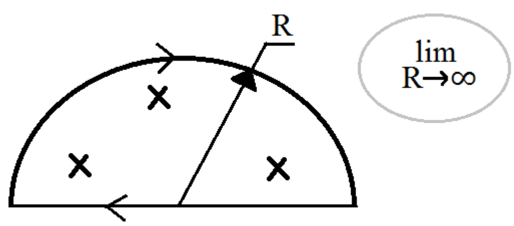
•Residue Theorem (V.2.2: p.112):
Let f be analytic in the region G except for the isolated singularities $a_1, a_2, \ldots, a_m$. If γ is a closed rectifiable curve in G which does not pass through any of the points a_k and if $\gamma \approx 0$ in G, then:

$$\frac{1}{2\pi i} \int_\gamma f = \sum_{k=1}^{m} n(\gamma; a_k) \cdot Res(f; a_k).$$

For definition of '≈' see p.95.

★ For poles of order m, we may calculate $\boldsymbol{Res(f; a) = \lim_{z \to a} (z-a)^m f(z)}$ (think Laurent Series of f).

• Handling Infinite Limits in Integrals (V: See p.113):
The trick seems to be to use an integral of a nice function close to the given one on a nice finite curve, apply the residue theorem to this new integral, algebraically solve for the integral that you want given a relationship to the nice integral, then take the limit as the finite curve goes to infinite radius.



- Rouche's Theorem (V: p.125):

Suppose f and g are *meromorphic* in a neighborhood of $\overline{B(a;R)}$ with no zeros or poles on $\partial B(a;R)$. If Z_f, Z_g (P_f, P_g) are the number of zeros (poles) of f and g inside γ counted according to their multiplicities and if $|f(z)+g(z)| \leq |f(z)|+|g(z)|$, then $\boldsymbol{Z_f - P_f = Z_g - P_g}$.

- Argument Principle (V: p.123):

Let f be meromorphic in G with poles $p_1, \ldots, p_m$ and zeros $z_1, \ldots, z_n$ counted according to multiplicity. If γ is a closed rectifiable curve in G with $\gamma \approx 0$ and not passing through $p_1, \ldots, p_m$, $z_1, \ldots, z_n$, then:

$$\boldsymbol{\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{k=1}^{n} n(\gamma; z_k) - \sum_{j=1}^{m} n(\gamma; p_j)}$$

- Max Mod Theorem (VI.1.1: p.128):

If G is a region and $f : G \to \mathbb{C}$ is analytic such that $\exists a \in G$ with $\forall z \in G, |f(z)| \leq |f(a)|$, then f is constant.