Čech Cohomology Groups & Riemann Surfaces

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0: Abstract

1: Sheaves

2: Čech Cohomology

1.) **Cochains** $c = (f_{i_0,\dots,i_n})$
2.) **Coboundary Operators** d^n
3.) **Cocycles** $\check{Z}^n(\mathscr{U},\mathfrak{F})$ **and Coboundaries** $\check{B}^n(\mathscr{U},\mathfrak{F})$; $d^n c = 0$ and $c = d^n c'$
(these are special types of cochains relative to the operators)

4.) **Cochain Complexes** $0 \to \check{C}^0(\mathscr{U},\mathfrak{F}) \xrightarrow{d^0} \check{C}^1(\mathscr{U},\mathfrak{F}) \xrightarrow{d^1} \check{C}^2(\mathscr{U},\mathfrak{F}) \xrightarrow{d^2} \dots$

5.) **Cohomology Groups** $\check{H}^n(\mathscr{U},\mathfrak{F}) = \check{Z}^n(\mathscr{U},\mathfrak{F}) / \check{B}^n(\mathscr{U},\mathfrak{F})$

6.) **Cohomology Groups in the Limit** $\check{H}^n(X,\mathfrak{F}) := \varinjlim \check{H}^n(\mathscr{U},\mathfrak{F})$

3. An Example

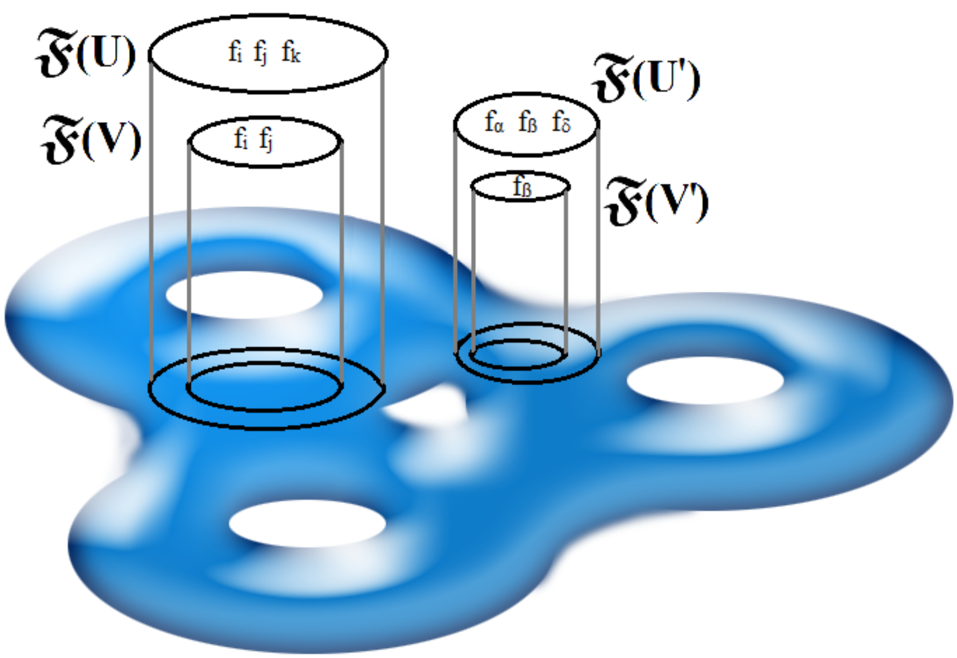
References

0: Abstract

The point of homology and cohomology theories is to assign algebraic invariants to topological spaces.

The goal of this project is to understand *a* co-homological construction created by *Eduard Čech*, in which the invariants are sequences of abelian groups.

The viewpoint will be guided by Rick Miranda's text [1] with help from other sources along the way. After the main theoretical constructions are accomplished, we will provide an enlightening example relevant to Riemann surface theory. Let's first define sheaves!



1: Sheaves

• Def: (p.269) For a topological space X, a **presheaf of groups $\mathfrak{F}$ on X** is a pair:

$$\left(\{\mathfrak{F}(U)\}_{U\subseteq X} \quad , \quad \{\rho_V^U : \mathfrak{F}(U) \to \mathfrak{F}(V)\}_{U,V\subseteq X}\right)$$

respectively, a collection of *groups* and *group homomorphisms* (**restriction maps** from U to V) which satisfy a **composition rule**:

$$\rho_W^U = \rho_W^V \circ \rho_V^U, \qquad \text{for } W \subseteq V \subseteq U.$$

The $f \in \mathfrak{F}(U)$ are referred to as **sections of $\mathfrak{F}$ over U** (or **global sections** in the special case when $U = X$).

As a technicality, we define $\mathfrak{F}(\emptyset) := \{e\}$ (trivial group). Note also that each group has the identity morphisms ρ_U^U.

• Def: (p.272) Given a presheaf $\mathfrak{F}$ on X, a subset $U \subseteq X$, and an open covering $\{U_i\}_{i\in I}$ of U. We say that $\mathfrak{F}$ satisfies the **sheaf axiom** for the subset U and the cover $\{U_i\}_{i\in I}$ if whenever one has f_i's in $\mathfrak{F}(U)$ which "agree on the intersections of the covering" i.e.:

$$\forall i, j \in I, \qquad \rho_{U_i\cap U_j}^{U_i}(f_i) = \rho_{U_i\cap U_j}^{U_j}(f_j)$$

then these sections patch together uniquely to give a section on U. In other words:

$$\exists!\ f \in \mathfrak{F}(U) \text{ such that: } \forall i \in I, \qquad \rho_{U_i}^U(f) = f_i.$$

In the event that $\mathfrak{F}$ satisfies the sheaf axiom for *every cover* of *every open subset* $U \subseteq X$, we say $\mathfrak{F}$ is a **sheaf**.

Note that replacing the word "group" above with other algebraic structures gives analogous definitions. We will need to start at least with a pre-sheaf of (abelian) groups for the purpose of creating quotient groups $\check{\mathrm{H}}^n$ as we will see.

It just so happens that the sheaf axiom can be encoded into the cohomology construction in the first node (at $\check{\mathrm{H}}^1$) as Miranda explains on (p.292). We leave this connection as an (Exercise) at the end.

2: Čech Cohomology

Given a presheaf of abelian groups $\mathfrak{F}$ on a topological space X. The outline for the cohomology construction is as follows:

Subsections:

1.) **Cochains** $c = (f_{i_0,\dots,i_n})$

2.) **Coboundary Operators** d^n

3.) **Cocycles** $\check{Z}^n(\mathscr{U}, \mathfrak{F})$ **and Coboundaries** $\check{B}^n(\mathscr{U}, \mathfrak{F})$; $d^n c = 0$ and $c = d^n c'$
(these are special types of cochains relative to the operators)

4.) **Cochain Complexes** $0 \to \check{C}^0(\mathscr{U}, \mathfrak{F}) \xrightarrow{d^0} \check{C}^1(\mathscr{U}, \mathfrak{F}) \xrightarrow{d^1} \check{C}^2(\mathscr{U}, \mathfrak{F}) \xrightarrow{d^2} \dots$

5.) **Cohomology Groups** $\check{H}^n(\mathscr{U}, \mathfrak{F}) = \check{Z}^n(\mathscr{U}, \mathfrak{F}) / \check{B}^n(\mathscr{U}, \mathfrak{F})$

6.) **Cohomology Groups in the Limit** $\check{H}^n(X, \mathfrak{F}) := \varinjlim \check{H}^n(\mathscr{U}, \mathfrak{F})$

<< 2.1: Cochains >>

Let $\mathfrak{F}$ be a presheaf of *abelian* groups on a topological space $\boldsymbol{X}$ throughout the discussion below.

• Def: (Notation): (p.291)

Let $\mathscr{U} = \{U_i\}_{i\in I}$ be an open covering of X, and fix an integer $n \geq 0$.

For every collection of indices $(i_0, i_1, \ldots, i_n)$, we sometimes denote the intersection of the corresponding open sets by:

$$U_{i_0,i_1,\ldots,i_n} := U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_n}.$$

When we get to the *coboundary* definition, we will need the notation for the deletion of one of the indices: "$\widehat{i_k}$". That is:

$$U_{i_0,i_1,\ldots,\widehat{i_k},\ldots,i_n} := U_{i_0,i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n}$$

• Def: (p.291) A **Čech** n**-cochain** (or simply n**-cochain**) for the presheaf $\mathfrak{F}$ over the open cover $\mathscr{U}$ is a *collection of sections of* $\mathfrak{F}$, denoted

$$(f_{i_0,\ldots,i_n}) := \{f_{i_0,\ldots,i_n}\}_{(i_0,\ldots,i_n)},$$

where we have one $f_{i_0,\ldots,i_n} \in U_{i_0,\ldots,i_n}$ for each multi-index $(i_0, \ldots, i_n)$ that yields a distinct intersection $U_{i_0,\ldots,i_n}$.

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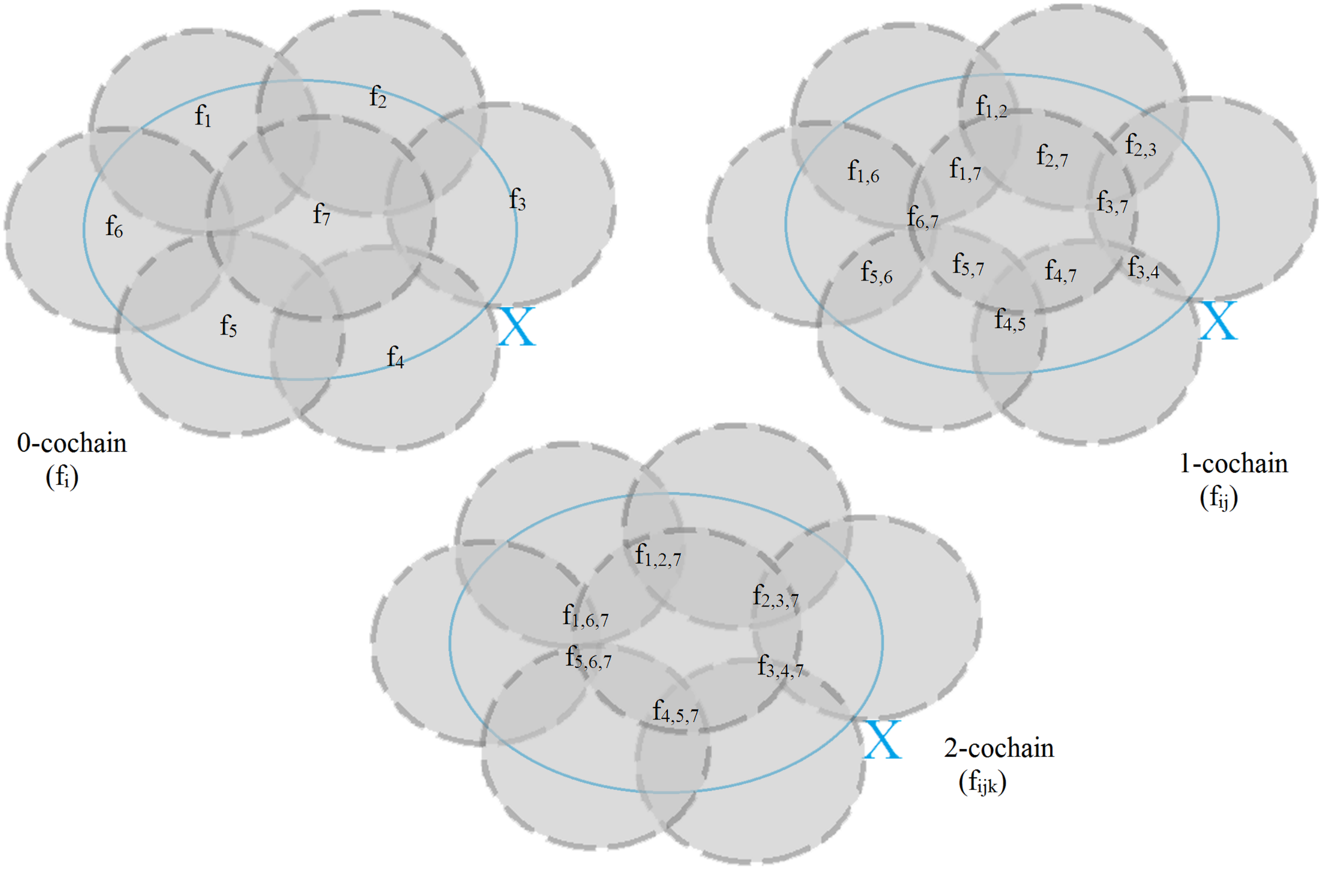
• Def: The **space of Čech** n**-cochains for** $\mathfrak{F}$ **over** $\mathscr{U}$ is denoted by:

$$\check{C}^n(\mathscr{U}, \mathfrak{F}) := \prod_{(i_0,i_1,\ldots,i_n)} \mathfrak{F}(U_{i_0,i_1,\ldots,i_n})$$

I.e. for each multi-index $(i_0, \ldots, i_n)$ in the collection, there is an associated *abelian group* $\mathfrak{F}(U_{i_0,\ldots,i_n})$ for which we have a choice of section $f_{i_0,\ldots,i_n}$ to extract. Note also via the direct product structure, $\check{C}^n(\mathscr{U}, \mathfrak{F})$ is an abelian group as well.

Visualizing n-Cochains in Low Index Size Cases:

A Čech **0-cochain** (n=0) is simply a collection ($f_i \in \mathfrak{F}(U_i)$); that is, one gives a section of $\mathfrak{F}$ over each open set in the cover. A **1-cochain** is a collection of sections of $\mathfrak{F}$ over every *double intersection* of open sets in the cover. "*...triple intersection...*", etc. See the figures:



<< 2.2: Coboundary Operators>>

• Def: (p.291) Define, for each $n \geq 0$, a map $\boldsymbol{d^n}$, called a **coboundary operator**:

$$\boldsymbol{d^n : \check{C}^n(\mathscr{U}, \mathfrak{F}) \to \check{C}^{n+1}(\mathscr{U}, \mathfrak{F})}$$

by setting:

$$\boldsymbol{d^n\big((f_{i_0,\ldots,i_n})\big) := (g_{j_0,\ldots,j_{n+1}})}$$

where

$$\boldsymbol{g_{j_0,\ldots,j_{n+1}} := \sum_{k=0}^{n+1} (-1)^k f_{j_0,\ldots,\widehat{j_k},\ldots,j_{n+1}}\big|_{U_1 \cap \ldots \cap U_{n+1}}}$$

where we are using our original sections from the n-cochain and restricting them to the new intersected sets.

Take away: Each d^n maps (n)-cochains to $(n+1)$-cochains in a contrived way.

Simple Cases Again:

At the **0-level**, d^0 sends a 0-cochain (f_i) to a 1-cochain (g_{ij}) where: $g_{ij} = f_j - f_i$. Of course each of these gets restricted to the appropriate $U_i \cap U_j$:

$$\boldsymbol{d^0\big((f_i)\big) = (f_j - f_i)}.$$

At the **1 level**, d sends a 1-cochain (f_{ij}) to the 2-cochain (g_{ijk}), where: $g_{ijk} = f_{jk} - f_{ik} + f_{ij}$, restricted to the $U_i \cap U_j \cap U_k$'s as before:

$$\boldsymbol{d^1\big((f_{ij})\big) = (f_{jk} - f_{ik} + f_{ij})}.$$

At the **2 level**, d sends a 2-cochain (f_{ijk}) to the 3-cochain:

$$\boldsymbol{d^2\big((f_{ijk})\big) = (f_{jkl} - f_{ikl} + f_{ijl} - f_{ijk})}.$$

restricted to the U_{ijkl}'s etc.

<< 2.3 Cocycles and Coboundaries >>

• Def: (p.292) Any *n-cochain* $\boldsymbol{c} := (f_{i_0,\ldots i_n})$ with $\boldsymbol{d^n c = 0}$ is called an **n-cocycle**.

The **space of n-cocycles** is denoted by $\check{\mathbf{Z}}^{n}(\mathscr{U}, \mathfrak{F})$.

• Def: Any *n-cochain* $\boldsymbol{c}$ which is in the image of d^{n-1} (coming out of the space of $(n-1)$-cochains) is called an **n-coboundary**.

The **space of n-coboundaries** is denoted by $\check{\mathbf{B}}^{n}(\mathscr{U}, \mathfrak{F})$

<< 2.4: Complexes >>

• Def: [5] "A **[co-]chain complex** is an algebraic structure that consists of a sequence of abelian groups (or modules) and a sequence of homomorphisms between consecutive groups such that the image of each homomorphism is included in the kernel of the next." Again:

$$\boldsymbol{Im(\varphi^i) \subseteq ker(\varphi^{i+1})}$$

Note that this differs from an exact sequence in that the exact sequence requires equality instead of inclusion.

• Def: (p.292) Since there are no (-1)-cochains, $\check{\mathrm{B}}^0(\mathscr{U}, \mathfrak{F}) = 0$ always. Then together with the fact that $d^n \circ d^{n-1} = 0$ (Exercise), we get the sequence of *spaces of Čech n-cochains*:

$$\mathbf{0} \to \check{\mathbf{C}}^{\mathbf{0}}(\mathscr{U}, \mathfrak{F}) \xrightarrow{d^0} \check{\mathbf{C}}^{\mathbf{1}}(\mathscr{U}, \mathfrak{F}) \xrightarrow{d^1} \check{\mathbf{C}}^{\mathbf{2}}(\mathscr{U}, \mathfrak{F}) \xrightarrow{d^2} \ldots$$

forms a cochain complex that we call the **Čech cochain complex**.

The failure or success of each node in the sequence to be exact can be measured by the cohomology groups (next). The sequence will be exact when each $\check{H}^n$ is zero.

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<< 2.5: Cohomology Groups >>

Recall $\mathfrak{F}$ is a presheaf of abelian groups on X.

★ Def: (p.292) The $\boldsymbol{n^{th}}$ **cohomology group of $\mathfrak{F}$ with respect to the cover** $\mathscr{U}$ is defined as:

$$\check{\mathrm{H}}^n(\mathscr{U}, \mathfrak{F}) := \check{\mathrm{Z}}^n(\mathscr{U}, \mathfrak{F}) \;/\; \check{\mathrm{B}}^n(\mathscr{U}, \mathfrak{F})$$

Read "n-cocycles modulo n-coboundaries".

Note that both of the components of the quotient are subgroups of $\check{\mathrm{C}}^n(\mathscr{U}, \mathfrak{F})$, since $\boldsymbol{ker(d^n)} = \check{\mathrm{Z}}^n(\mathscr{U}, \mathfrak{F})$ and $\boldsymbol{Im(d^{n-1})} = \check{\mathrm{B}}^n(\mathscr{U}, \mathfrak{F})$ and the d^i's are homomorphisms. And by our choice of presheaf $\mathfrak{F}$, every subgroup in sight is abelian, so the quotient is itself a group first off, but it is also abelian.

<< 2.6: Cohomology Groups in the Limit >>

• Def: Fix a presheaf of abelian groups $\mathfrak{F}$ on X and an integer $n \geq 0$.
The $\boldsymbol{n^{th}}$ **Čech cohomology group of $\mathfrak{F}$ on $\boldsymbol{X}$** is the group:

$$\check{\mathrm{H}}^n(X, \mathfrak{F}) := \varinjlim \check{\mathrm{H}}^n(\mathscr{U}, \mathfrak{F})$$

where the *direct limit* is with respect to $\mathscr{U}$.

This last definition is the most important one regarding the goal of cohomology, however it is also more involved than I have time for given the due date. So we'll defer this one to Miranda (pp.293-296). These limiting groups are independent of the cover chosen and are hence intrinsic to the topological space and the given presheaf.

At this point we are done with the theory. Let's observe next...

3: An Example

The following is Problem "M" from the end of Section IX.3 (p.301) of Miranda.

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Let X be the Riemann Sphere $\mathbb{C}_\infty$, and let $U_0 = X - \{0\}$ and $U_1 = X - \{\infty\}$ be the standard open covering $\mathscr{U}$ of X. Compute $\check{\mathrm{H}}^1(\mathscr{U}, \mathcal{O}_X[n \cdot \infty])$ for all n explicitly by writing down the spaces of relevant cochains, computing the 1-cocycles and 1-coboundaries, and taking the quotient group.

Preliminaries: First we need to define what exactly $\mathcal{O}_X[n{\cdot}\infty]$ is. On (p.271) we have:

"For a Riemann surface X and D a *divisor* on X, then $\mathcal{O}_X[D](U)$ is the set of all *meromorphic functions* on U which satisfy the condition that:

$$\boldsymbol{ord_p(f) \geq -D(p) \qquad \forall p \in U}.$$

This gives a presheaf of [abelian] groups on X (Exercise), call it $\mathcal{O}_X[D]$ (or simply $\mathcal{O}[D]$). Note that $\mathcal{O}_X[0]$ is exactly the presheaf $\mathcal{O}_X$ of holomorphic functions on X."

Then on (p.129) we get "for a compact Riemann surface, the set of all divisors on X, denoted $Div(X)$ is the free abelian group on the set of points of X. And we write:

$$\boldsymbol{D : X \to \mathbb{Z}; \qquad D = \sum_{p \in X} D(p) \cdot p}$$

where the set of points p such that $D(p) \neq 0$ is discrete."

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(Next page)

Solution:

With this notation defined, say we fix an $n \geq 0$, then $D := n \cdot \infty$ just means that the value of the function $D : \mathbb{C}_\infty \to \mathbb{Z}$ is zero everywhere except at the point ∞ in which it has the value n.

Interpreting the local groups now for the presheaf, we have that $\mathcal{O}_X[n \cdot \infty](U_0)$ is the set of all mero's on $U_0 = \mathbb{C}_\infty - \{0\}$ which satisfy:

$$ord_p(f) \geq \begin{cases} -n, & \text{at } \infty \\ 0, & \text{otherwise} \end{cases}$$

and

$\mathcal{O}_X[n \cdot \infty](U_1)$ is the set of all mero's on $U_1 = \mathbb{C}_\infty - \{\infty\}$ which satisfy:

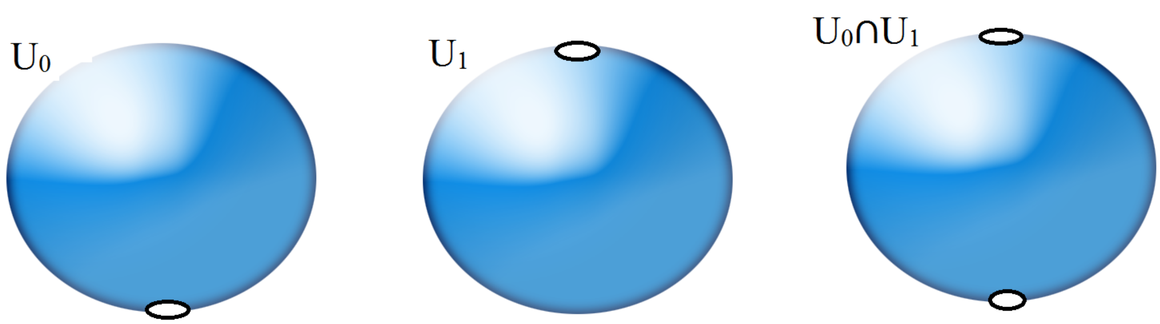
$$ord_p(f) \geq 0 \qquad (\text{since } \infty \notin U_1)$$

That is,

$\boldsymbol{\mathcal{O}_X[n \cdot \infty](U_0) = \{}$mero's on $\boldsymbol{U_0}$ having at most one pole with $\boldsymbol{ord_\infty(f) \geq -n\}}$

and

$\boldsymbol{\mathcal{O}_X[n \cdot \infty](U_1) = \{}$holo's on $\boldsymbol{U_1\}}$



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From this we get the space of 0-cochains, $\check{C}^0(\mathscr{U}, \mathcal{O}_X[\mathcal{D}])$ is comprised of choices of pairs of appropriate functions on U_0 and U_1. And the space of 1-cochains, $\check{C}^1(\mathscr{U}, \mathcal{O}_X[\mathcal{D}])$ is comprised of single choices of holo's on $U_0 \cap U_1$.

(Continues)

Recall we just found $C^0 := \check{\mathrm{C}}^0(\mathscr{U}, \mathcal{O}_{\mathcal{X}}[\mathcal{D}])$ and $C^1 := \check{\mathrm{C}}^1(\mathscr{U}, \mathcal{O}_{\mathcal{X}}[\mathcal{D}])$.
We want to use this to find $\mathrm{H}^1 := \check{\mathrm{H}}^1(\mathscr{U}, \mathcal{O}_X[D])$.

The way we do this is by identifying who $Z^1 := \check{\mathrm{Z}}^1(\mathscr{U}, \mathcal{O}_{\mathcal{X}}[\mathcal{D}])$ and $B^1 := \check{\mathrm{B}}^1(\mathscr{U}, \mathcal{O}_{\mathcal{X}}[\mathcal{D}])$ are and then we compute their quotient.

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By definition, we have the 1-cocycles are just 1-cochains such that $d^1 c = 0$. And the 1-coboundaries are the 1-cochains that are hit by a 0-cochain across d^0. That is $c = d^0 c'$ schematically.

With the help of our work on (p.7), these conditions become formulaic.

Particularly, we found that:

$$\boldsymbol{d^0((f_i)) = (f_j - f_i) \equiv (g_{ij})} \qquad \text{and} \qquad \boldsymbol{d^1((f_{ij})) = (f_{jk} - f_{ik} + f_{ij}) \equiv (g_{ijk})},$$

where $\boldsymbol{i, j, k \in \{0, 1\}}$ in our case corresponding to neighborhoods U_0 and U_1.

Starting with the first formula, noting that there is only 1 possible intersection, namely $U_0 \cap U_1$, we can take the 1-cochain $(f_j - f_i) = \{f_1 - f_0\}$ for a holomorphic function f_1 on U_1 and a meromorphic f_0 on U_0 both restricted to the intersection $U_0 \cap U_1$ (where they are both holomorphic by definition). Thus elements of the image of d^0, i.e. elements of B^1 are just holomorphic functions on $U_0 \cap U_1$ (viewed as 1-cochains).

Now, since there is only one choice available for 1-cochain, namely a holomorphic map on $U_0 \cap U_1$ and no triple intersections available, we have the second equation yields: $\{f_{01} - f_{01} + f_{01}\} = \{0\}$ (the zero-2-cochain)–which gives the zero function for each multi-index.

$\implies f_{01} = 0$. So finally we found that the only element of Z^1 is the zero-1-cochain. By inclusion, there must only be the zero-1-cochain in B^1 as well (this is fine since 0 is a holomorphic function technically).

$\therefore H^1 = Z^1/B^1 = \{\text{zero-1-cochain}\}/\{\text{zero-1-cochain}\} \cong 0$

So we conclude that $\boldsymbol{\check{\mathrm{H}}^1(\mathscr{U}, \mathcal{O}_{\mathbb{C}_\infty}[n \cdot \infty]) \cong 0}$ ■

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Closing Statements:

The previous example showed us how complicated things can get in the easiest of cases. But, it also implicitly spelled out how to compute the Z^n and B^n in general using the formulas provided by the coboundary operators, given knowledge about the cover and the amount of intersections at each level... and knowledge of the spaces of cochains: C^{n-1} and C^n. Good hunting!

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Challenges/Exercises:

1.) Compute $d^3\big((f_{ijkl})\big)$ via the formula given in section 2.2.

2.) Look into why the formula was defined the way it was.

3.) Prove that $d^n \circ d^{n-1} = 0$ for any $n \geq 1$ [Hint: A proof is given in [6]].

4.) In the example, prove that $\mathcal{O}_X[D]$ gives a presheaf of abelian groups.

5.) Show that a presheaf on X forms a category. Alternatively, show that a presheaf forms a contravariant functor.

6.) Find an example of a computation where $\check{\mathrm{H}}^n$ is not 0.

7.) Investigate the connection between homology and cohomology. [Hint: There is a good youtube video out there for this one.]

8.) Chase down the definitions of: refinements of covers, sheaf maps and associated induced maps, direct limits of systems of groups etc needed for the limiting construction described on (pp.293-296 [1]).

9.) Explain how the sheaf condition emerges in the cohomology construction [Hint: See (p.292 [1])].

References

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