Theory of Tensors

By: Kevin Smith

Summer 2020 - Fall 2021

>> Read Me

Table of Contents:

Introduction

This document aims at dealing with tensor-field equations in the context of differential geometry and physics. The tensors we deal with are real-valued mappings that are linear in each variable. The variables can be vectors or dual vectors (or both, depending on the valence). Tensor fields are the upgraded version to manifolds, whereby the tensor components become dependent on location. Two tensors are equal if all their components are equal (similar to vector equality).

That being said, there is a lot that goes into these statements formally speaking and as one will see, the equations that arise are usually some type of differential or integral equation involving more advanced notions of the ones seen in calculus. Things like arc length, curvature, volume, and orientation are generalized. Technical tools for manipulating tensors are given and the culminating topics are *Geodesic Equations* and *Einstein's Field Equations*.

Out of all the texts I've come across on the subject, John Lee has the best in my opinion. William Boothby has a rivaling text I've enjoyed as well. Robert Wald's text covers a lot of content overlapping with Lee's Curvature book but with heavy application to Physics. For Complex Manifolds and Riemann Surfaces, Rick Miranda is my go to. See references for these and more!

Acknowledgements

I would like to thank my professors: Arthur Moore at OCC (290H), Jeffrey Streets (218AB) and Xiangwen Zhang (218C) at UCI, Julien Paupert (501) and Brett Koschwar (502) at ASU, and John Brevik (566) at CSULB for my progress in understanding this material.

Overall Section Citations

I.1: Chapter 12 [7].
I.2: Chapter 1 [7]; Chapter 1 [9]
I.3: Chapters 3, 10, 11, and 12 [7]

II.1: Chapter 2 and Appendix C [7]
II.2: Chapters 3, 8, 11, and Appendix C [7]
II.3: Chapters 4 and 5 [6]; Chapter 13 [7], Chapter 6 [8]

III.1: Chapters 12 and 14 [7]; Chapter 2.15 [2]
III.2: Chapters 14, 15, and 16 [7], Ch.3 [12], Ch.6 [11], Ch.6 [10] (Analysis), [1] (Measure)
III.3: Chapters 1, 14, and 16 [7]

IV.1: Chapters 4-7 [8], Chapter VII [11]
IV.2: Chapters 4-7 [8], Chapter VII [11], and a Wikipedia Article [15]
IV.3: Chapters 4-8 [8], Chapter VIII [11], [15], Section 13.3 of [13], and Ch.6-7 [14].

Appendix A: Developed from experience especially writing this document.
Appendix B: Chapter 12 (p.307-311)[7], Ch.10.4 (p.359-375) [3], and Ch.13/14 (p.340-377; p.387-388)[14].

<< Chapter I : Tensor Fields on Manifolds >>

This chapter introduces the main players in differential geometry.

1. Construction of Tensors (On A Vector Space)

Import Linear Algebra. Let V be an *n-dimensional vector space over* $\mathbb{R}$ with basis $\{\partial_1, ..., \partial_n\}$.

• Def: We define the **dual space**, denoted V^*, to be the set of linear functionals over V. That is,

$$V^* := \{\omega : V \to \mathbb{R} \mid \omega \text{ is linear}\}.$$

We also call the elements $\omega \in V^*$, **dual vectors** or **co-vectors**.

[Exercise: Prove V^* forms a vector space and show the *component functions* $\partial^i : v \mapsto v^i$ form a basis. Note that $\partial^i(\partial_j) = \delta^i_j$. The symbol δ^i_j is known as the **kronecker delta** and has value 1 if indices match and zero otherwise.]

• Def: The **double dual space**, denoted V^{**}, is just the dual of the dual of V. That is,

$$V^{**} := \{f : V^* \to \mathbb{R} \mid f \text{ is linear}\}.$$

[Exercise: Show the map $v \mapsto (f_v : \omega \mapsto \omega(v))$ gives a vector space isomorphism between V and V^{**}. So that we can identify v and f_v and write: $v(\omega) := \omega(v)$.]

★ Def: Let us define the following related vector spaces (for each $k, l \in \mathbb{N}$):

$$\mathcal{T}^k_l(V) := \left\{ \varphi : (V^*)^k \times V^l \to \mathbb{R} \;\middle|\; \varphi \text{ is linear in each variable} \right\},$$

whose elements we call **multi-linear functionals** or **(k.l)-tensors**. In this new notation:

$$\mathcal{T}^1_0(V) := V \qquad \text{and} \qquad \mathcal{T}^0_1(V) := V^*$$

★ Def: Given a pair of **tensor valences** (k, l) and (m, n), we have a map:

$$\mathcal{T}^k_l(V) \times \mathcal{T}^m_n(V) \to \mathcal{T}^{k+m}_{l+n}(V); \qquad (\varphi, \psi) \mapsto \varphi \otimes \psi, \text{ where}$$

$$\left[\varphi \otimes \psi\right](\omega^1, ..., \omega^{k+m}, v_1, ..., v_{l+n})$$
$$:= \varphi(\omega^1, ..., \omega^k, v_1, ..., v_l) \cdot \psi(\omega^{k+1}, ...\omega^{k+m}, v_{l+1}, ..., v_{l+n}).$$

We call $\otimes$ the **tensor product of φ and ψ**.

We just displayed the tool for constructing mixed tensors on a vector space from the base cases of $(k, l) = (1, 0)$ and $(0, 1)$, i.e. from vectors and dual vectors! Let's observe a simple case just for clarity:

Example: Given a vector $\boldsymbol{v}$ and a covector $\boldsymbol{\omega}$, we define the action of $\boldsymbol{v} \otimes \boldsymbol{\omega}$ on $\boldsymbol{V}^* \times \boldsymbol{V}$ via:

$$[\boldsymbol{v} \otimes \boldsymbol{\omega}](\boldsymbol{\alpha}, \boldsymbol{x}) := \boldsymbol{v}(\boldsymbol{\alpha}) \cdot \boldsymbol{\omega}(\boldsymbol{x}) \in \mathbb{R}$$

where of course $\boldsymbol{\alpha} \in \boldsymbol{V}^*$ and $\boldsymbol{x} \in \boldsymbol{V}$ are arbitrary.

Note: Since multiplication in $\mathbb{R}$ is commutative, we may identify rearrangements of the factors of a tensor and their arguments. So that one may write for example:

$$\boldsymbol{\varphi} \otimes \boldsymbol{\psi}(\boldsymbol{v}, \boldsymbol{\omega}) = \boldsymbol{\varphi}(\boldsymbol{v})\boldsymbol{\psi}(\boldsymbol{\omega}) = \boldsymbol{\psi}(\boldsymbol{\omega})\boldsymbol{\varphi}(\boldsymbol{v}) = \boldsymbol{\psi} \otimes \boldsymbol{\varphi}(\boldsymbol{\omega}, \boldsymbol{v})$$

and in general, not necessarily have all the contra- and co- factors grouped together in the display.

• Prop: Given a vector space $\boldsymbol{V}$ with basis $\{\boldsymbol{\partial}_1, ..., \boldsymbol{\partial}_n\}$ and dual basis $\{\boldsymbol{\partial}^1, ..., \boldsymbol{\partial}^n\}$, we have the **basis for $\mathcal{T}_l^k(\boldsymbol{V})$** given by:

$$\left\{ (\boldsymbol{\partial}_{i_1} \otimes ... \otimes \boldsymbol{\partial}_{i_k}) \otimes (\boldsymbol{\partial}^{j_1} \otimes ... \otimes \boldsymbol{\partial}^{j_l}) \;\middle|\; i_a, j_b \in \{1, ..., n\} \right\}$$

so that $\boldsymbol{dim}(\mathcal{T}_l^k(\boldsymbol{V})) = \boldsymbol{n}^{k \cdot l}$.

Proof: [Exercise: Prove the linear independence and spanning properties.]

• Def: With this basis, we can expand a given $\boldsymbol{\varphi} \in \mathcal{T}_l^k(\boldsymbol{V})$, via:

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}^{i_1 ... i_k}_{j_1 ... j_l} \boldsymbol{\partial}_{i_1} \otimes ... \otimes \boldsymbol{\partial}_{i_k} \otimes \boldsymbol{\partial}^{j_1} \otimes ... \otimes \boldsymbol{\partial}^{j_l},$$

(using the **Einstein summation convention**). The **tensor components** can be found via:

$$\boldsymbol{\varphi}^{i_1 ... i_k}_{j_1 ... j_l} := \boldsymbol{\varphi}(\boldsymbol{\partial}^{i_1}, ..., \boldsymbol{\partial}^{i_k}, \boldsymbol{\partial}_{j_1}, ..., \boldsymbol{\partial}_{j_l})$$

since as we've seen, we just get the kronecker delta a bunch of times going to 1. The collection of all a tensor's components forms a *multi-dimensional array* (think $\boldsymbol{\varphi}[\boldsymbol{i_1}]...[\boldsymbol{i_k}][\boldsymbol{j_1}]...[\boldsymbol{j_l}]$ in Java).

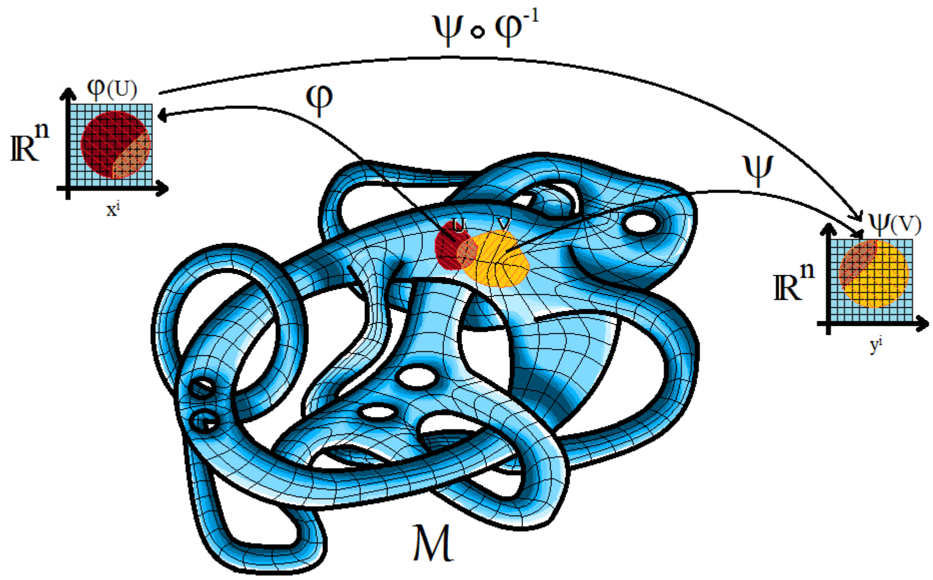
Exercise:
a.) Take the standard basis in $\boldsymbol{V} = \mathbb{R}^2$ and find the component $\boldsymbol{\varphi}^1_2$, where

$$\boldsymbol{\varphi} := \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes [1 \;\; 0] - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes [0 \;\; 1] - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes [1 \;\; 0] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes [0 \;\; 1].$$

b.) A tensor is called **simple** if it can be written as a tensor product of vectors and covectors. Prove the above $\boldsymbol{\varphi}$ is simple using bilinearity. Find a non-simple tensor of the same valence.

Lastly, our notion of tensor as a multi-linear map $\boldsymbol{\varphi} : (\boldsymbol{V}^*)^k \times \boldsymbol{V}^l \to \mathbb{R}$ can be abstracted in a few ways. See the Appendix B for more.

2. Manifolds and Coordinate Systems



Import Topology. Real manifolds are topological objects, that can be observed locally through charts into $\mathbb{R}^{\boldsymbol{n}}$ for some fixed dimension $\boldsymbol{n}$. Due to necessity, they are axiomatized as follows:

• <u>Def:</u> (p.3 [7]) A **topological n-manifold** is a topological space $\boldsymbol{\mathcal{M}} = (\boldsymbol{M}, \boldsymbol{T} \subseteq \boldsymbol{\mathcal{P}}(\boldsymbol{M}))$ with the following three properties:

(i) $\boldsymbol{\mathcal{M}}$ is a *Hausdorff* space (any two points have distinct neighborhoods around them).

(ii) $\boldsymbol{\mathcal{M}}$ has a countable basis for its topology $\boldsymbol{T}$ (i.e. is *Second-Countable*), and

(iii) At each point $\boldsymbol{p} \in \boldsymbol{\mathcal{M}}$ there exists a neighborhood containing p that is homeomorphic to a subset of $\mathbb{R}^{\boldsymbol{n}}$ for a fixed $\boldsymbol{n}$.

As Lee mentions on page 3, some motivations for (i) and (ii) are that we get uniqueness of limits of convergent sequences and we get the existence of *partitions of unity*, which among other things allows us to define derivatives and integrals of *differential forms* and gives us global constructs such as a *Riemannian metric*, built from local definitions. [Exercise: Explore the necessity of (i) and (ii).]

Let's focus on (iii) next. But first, we reassign the symbols $\boldsymbol{\varphi}, \boldsymbol{\psi},$ etc. for charts instead of tensors on a single vector space. We will be upgrading them to capitol greek letters for tensor fields later.

(*Continues*)

The following is based off of (p.4-15 [7]).

• Def: Item (iii) says $\forall \boldsymbol{p} \in \boldsymbol{\mathcal{M}}$, there exists a neighborhood, $\boldsymbol{U} \in \boldsymbol{T}$, containing $\boldsymbol{p}$ and there exists a homeomorphism $\boldsymbol{\varphi} : \boldsymbol{U} \to \mathbb{R}^{\boldsymbol{n}}$.

We call the pair $(\boldsymbol{U}, \boldsymbol{\varphi})$ a **chart on $\boldsymbol{\mathcal{M}}$**. We may refer to $\boldsymbol{U}$ as the **coordinate neighborhood** and $\boldsymbol{\varphi}$ as the **chart map**. The image of the chart map, $\boldsymbol{\varphi}(\boldsymbol{U})$, is called a **coordinate system**, since in a basis, $\boldsymbol{\beta}$ for $\mathbb{R}^{\boldsymbol{n}}$, we can assign coordinates $(\boldsymbol{x}^1, ..., \boldsymbol{x}^n)$ to each point, where $\forall \boldsymbol{q} \in \boldsymbol{U},\ \boldsymbol{x}^i_{\boldsymbol{q}} := [\boldsymbol{\varphi}(\boldsymbol{q})]^i_{\boldsymbol{\beta}}$. If it is the case that $[\boldsymbol{\varphi}(\boldsymbol{p})]_{\boldsymbol{\beta}} = (\boldsymbol{0}, ..., \boldsymbol{0})$, we say the chart is **centered at $\boldsymbol{p}$** and denote it by $(\boldsymbol{U}, \boldsymbol{\varphi})_{\boldsymbol{p}}$.

Note that we can post-compose with another homeomorphism, say $\boldsymbol{\rho} : \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}^{\boldsymbol{n}}$, if we need to make adjustments (say to the center point or the shape of the image), yielding $(\boldsymbol{U}, \boldsymbol{\rho} \circ \boldsymbol{\varphi})$ as a new chart since composition preserves homeomorphisms.

• Def: If $(\boldsymbol{U}, \boldsymbol{\varphi})$ is a chart, then taking an open subset $\boldsymbol{V} \subseteq \boldsymbol{U}$ yields a new such pair $(\boldsymbol{V}, \boldsymbol{\varphi}|_{\boldsymbol{V}})$, called a **subchart**.

• Def: A choice of chart (without further qualification) for every point in the manifold is called a (topological) **atlas**. We may denote such an atlas by:

$$\boldsymbol{\mathcal{A}} = \bigcup_{\boldsymbol{p} \in \boldsymbol{\mathcal{M}}} (\boldsymbol{U}, \boldsymbol{\varphi})_{\boldsymbol{p}}.$$

If $\boldsymbol{q} \in \boldsymbol{\mathcal{M}}$ is contained in a chart for another point $\boldsymbol{p}$, we can just use $(\boldsymbol{U}, \boldsymbol{\varphi})_{\boldsymbol{p}}$ to assign coordinates around $\boldsymbol{q}$ (using a subchart say). That way, ultimately a (topological) atlas is specified by a cover of $\boldsymbol{\mathcal{M}}$, where each open set in the cover is centered arbitrarily.

We now proceed to define different regularity classes of structures on the manifold using atlases.

• Def: Given two charts $(\boldsymbol{U}, \boldsymbol{\varphi})$ and $(\boldsymbol{V}, \boldsymbol{\psi})$ with $\boldsymbol{U} \cap \boldsymbol{V} \neq \emptyset$, the fact that the chart maps are homeomorphisms gives existence of inverses and hence there exist two maps:

$$\boldsymbol{\psi} \circ \boldsymbol{\varphi}^{-1} : \boldsymbol{\varphi}(\boldsymbol{U} \cap \boldsymbol{V}) \to \boldsymbol{\psi}(\boldsymbol{U} \cap \boldsymbol{V}) \qquad \text{and} \qquad \boldsymbol{\varphi} \circ \boldsymbol{\psi}^{-1} : \boldsymbol{\psi}(\boldsymbol{U} \cap \boldsymbol{V}) \to \boldsymbol{\varphi}(\boldsymbol{U} \cap \boldsymbol{V})$$

called **transition maps** between the two coordinate systems – both contained in $\mathbb{R}^{\boldsymbol{n}}$!

• Def: Two charts are said to be **$\boldsymbol{C^k}$-compatible** if either $\boldsymbol{U} \cap \boldsymbol{V} = \emptyset$ (they have disjoint domains) or otherwise both transition maps are of regularity class $\boldsymbol{C^k}$. That is, for example $\boldsymbol{C^0}$ is continuous, $\boldsymbol{C^1}$ continuously differentiable, $\boldsymbol{C^k}$ is for $\boldsymbol{k}$-times differentiable with continuous $\boldsymbol{k^{th}}$ derivative, $\boldsymbol{C^\infty}$ is infinitely differentiable (a.k.a. **smooth**), $\boldsymbol{C^\omega}$ is real analytic, etc.. in the case of *complex manifolds*, their transition maps can be required to be *holomorphic* (see Ch.1 of [9]). What is required depends on what you are doing with the manifolds.

(Continues)

• Def: A collection of charts that cover $\mathcal{M}$ and are all mutually C^k-compatible is called a **C^k-atlas**. The special case of C^∞ is called a **smooth atlas**.

- -

Special Notes:

1.) We can always append an atlas with another chart that is mutually compatible with all existing members – to create a new atlas. This leads to the notion of **maximal atlases** (ones that are closed under containment of all possible compatible charts). Unfortunately, *Zorn's Lemma* is required for existence of these beasts.

2.) Chart compatibility is not an equivalence relation (transitivity fails), but atlas compatibility is (two atlases are compatible if all their charts are mutually compatible).

3.) Two atlases need not be compatible (they can be contained in different maximal atlases).

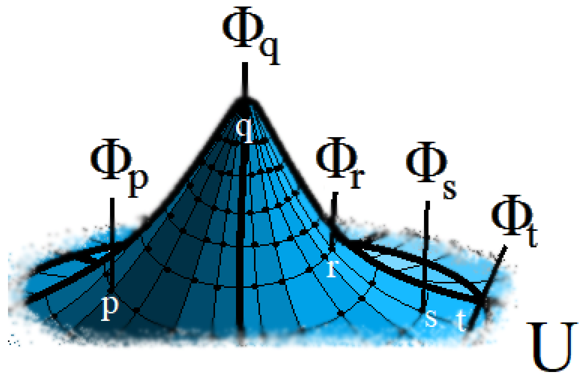
[Exercise: Explore (1) and (2), define a partial order structure on the set of all atlases on $\mathcal{M}$, look up Zorn's Lemma and see that it can be applied. Use maximal atlases to prove transitivity of atlas compatibility. Bonus: Disprove transitivity for chart compatibility, I never got that one!].

- -

• Def: In light of the special notes above, let us define a **C^k-structure** as such a maximal atlas $\mathcal{A}$ or equivalence class of atlases $[\mathcal{A}]$. Same goes for a **smooth structure**. A **C^k-manifold** or **smooth manifold** is a choice of appropriate structure, in addition to the manifold. One can denote this by $(\mathcal{M}, \mathcal{A})$.

Let us assume from now on that a particular smooth structure $(\mathcal{M}, \mathcal{A})$ is given. Our main concern now is to define tensor fields on these manifolds - the regularity of these objects will be determined by their component functions in coordinates.

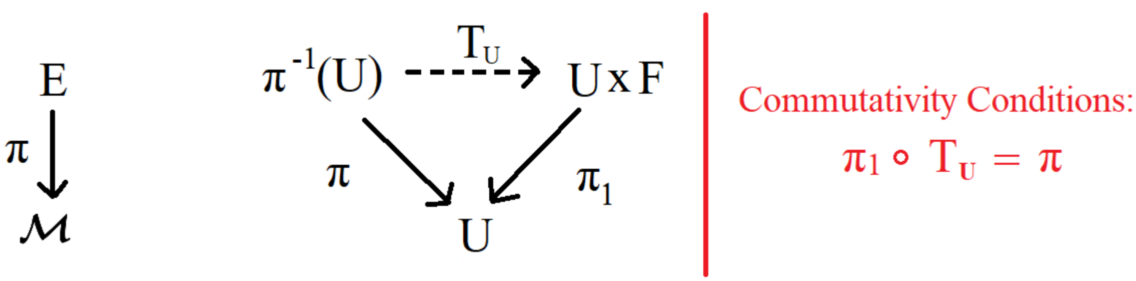
3. Fiber Bundles, Tensor Fields (On Manifolds), and Frames



The figure above gives an intuitive picture of a tensor field. At each point on the manifold, we have an associated tensor (the collection of which is declared by what's called a *section of a tensor bundle*). We proceed to define these terms and some associated notions. The following is based on Chapters 3,10,11, and 12 of [7].

We start with the most general notion of a fiber bundle (for reference).

• Def: (p.268 [7]) Let $\mathcal{M}$ and F be topological spaces. A **fiber bundle over $\mathcal{M}$ with model fiber F** is a topological space E, together with a surjective continuous map $\pi : E \to \mathcal{M}$ with the property that for each $p \in \mathcal{M}$, there exists a neighborhood U of p in $\mathcal{M}$ and a homeomorphism $T_U : \pi^{-1}(U) \to U \times F$, called the **local trivialization of E over U**, such that the following diagrams commute for all U:



The space E is called the **total space of the bundle**, $\mathcal{M}$ is its **base**, and π is its **projection**. "Above" each point, we have a topological space $E_p := \pi^{-1}(p)$ modeled by F.

• Def: (p.249 [7]) A **(real) vector bundle of rank k over $\mathcal{M}$** is a fiber bundle whose model fiber is a k-dimensional vector space. That is, the fibers $\pi^{-1}(p) =: E_p$ are all vector spaces of dimension k, modeled of course by $F = \mathbb{R}^k$. We also require $T_U|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ to be linear (on top of being a homeomorphism, this makes it a vector space isomorphism – in general there should be a structure preserving condition like this depending on the structure of F).

(Continues)

By Props 10.4 (p.252), 11.9 (p.276), and Exercise 12.18 (p.317) of [7], we have the *tangent, cotangent,* and *tensor bundles* (defined below) form vector bundles of ranks $\boldsymbol{n}$, $\boldsymbol{n}$, and $\boldsymbol{n^{k \cdot l}}$ respectively. Let us defer the definition of *tangent space at a point*, $\boldsymbol{T_p\mathcal{M}}$, until Section II.2. For now, think of it as a space of vectors eminating from the point $\boldsymbol{p}$ on the manifold.

• Def: (p.65) **(Tangent Bundles)**

$$T\mathcal{M} := \coprod_{p\in\mathcal{M}} T_p\mathcal{M}$$

• Def: (p.276) **(Cotangent Bundles)**

$$T^*\mathcal{M} := \coprod_{p\in\mathcal{M}} (T_p\mathcal{M})^*$$

• Def: (p.316) **((k,l)-Tensor Bundles)**

$$\mathcal{T}_l^k(T\mathcal{M}) := \coprod_{p\in\mathcal{M}} \mathcal{T}_l^k(T_p\mathcal{M})$$

That is, each of these "bundles" are disjoint unions of vector spaces indexed by $\boldsymbol{p \in \mathcal{M}}$, with the asterisk denoting the dual as we've seen. These bundles also have the structure of manifolds (which we'll need to talk about regularity). See the "Vector Bundle Chart Lemma" (p.253).

• Def: (p.255) Given a vector bundle $\boldsymbol{\pi : E \to \mathcal{M}}$, a **(local) section** of $\boldsymbol{\pi}$ is (without further qualification) a continuous map $\boldsymbol{\sigma : U \to E}$ such that $\boldsymbol{\pi \circ \sigma = Id_U}$ – that is, just a right inverse of the projection restricted above $\boldsymbol{U}$. A **(global) section** is just one where $\boldsymbol{U = \mathcal{M}}$.

• Def: We define **vector fields**, **covector fields**, and **tensor fields** to be local or global sections of their associated bundle projections. The **sets of all sections** of the above bundles form infinite-dimensional vector spaces over $\mathbb{R}$ and modules over $\boldsymbol{C^\infty(\mathcal{M})}$ [Exercise: Prove this!]. They are denoted by:

$$\Gamma(T\mathcal{M}),\ \Gamma(T^*\mathcal{M}),\ \text{and } \Gamma(\mathcal{T}_l^k(T\mathcal{M})).$$

• Def: (p.257) A **(local or global) frame** on a manifold is simply a collection of (local or global) sections, $\boldsymbol{\{\sigma_1, ..., \sigma_m\}}$, such that in the fiber above each point, $\boldsymbol{E_p}$, we have a basis given by: $\boldsymbol{\{\sigma_1(p), ..., \sigma_m(p)\}}$.

Notes: The $\boldsymbol{\sigma_i}$'s above are abstract symbols to be replaced by the appropriate field symbols. E.g. $\boldsymbol{\{X_1, ..., X_n\}}$, $\boldsymbol{\{\omega^1, ..., \omega^n\}}$, or $\boldsymbol{\{{}^1\Phi, ..., {}^m\Phi\}}$, etc. We usually write tensor fields expanded in a *coordinate frame*, $\boldsymbol{\{\partial_1, ..., \partial_n\}(p)}$, above a neighborhood (that is, locally), by:

$$\Phi = \Phi^{i_1 \dots i_k}_{j_1 \dots j_l}(p) \cdot \partial_{i_1}(p) \otimes ... \otimes \partial_{i_k}(p) \otimes \partial^{j_1}(p) \otimes ... \otimes \partial^{j_l}(p)$$

The point can be suppressed when it is clear. These coordinate frames are given by the *pushforwards* or *pullbacks* respectively from the standard frames in the charts. We will cover both of these notions in Section II.2.

<< Chapter II : Basic Manipulations in Coordinates >>

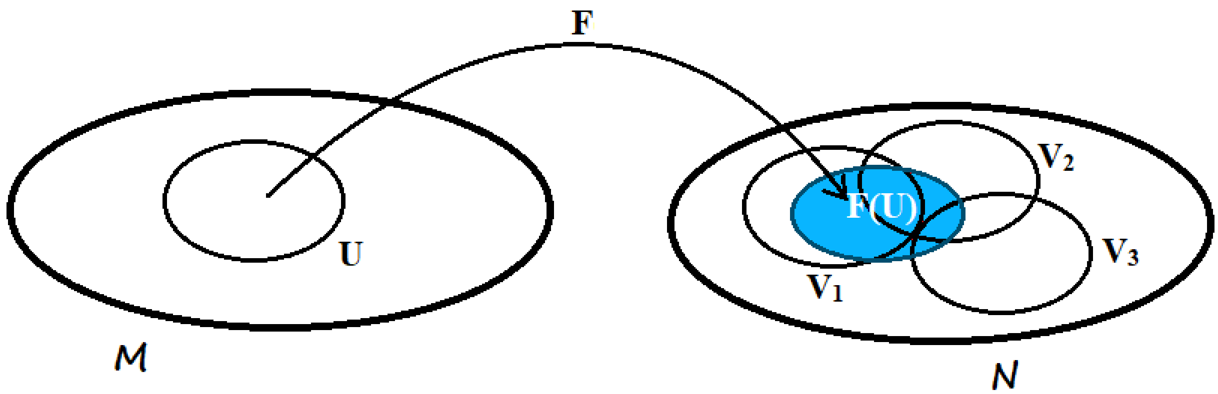
This chapter lists some things we can do with tensor fields: loosely speaking, we can use maps to induce fields on other manifolds, extract array data from them, put two fields together in certain ways, change valence, etc. First things first:

1. Mappings of Manifolds and Coordinate Representations

The reference for this section is Ch.2 [7].

• Def: At the base level, **mappings of manifolds $F : \mathcal{M} \to \mathcal{N}$** are just set maps. However, given the topologies of $\mathcal{M}$ and $\mathcal{N}$, we can talk about **continuous mappings of manifolds** by pullbacks of open sets being open (the usual topological definition). Note that the manifolds can have different dimensions.

If we want to talk about further regularity, we need to involve charts to ultimately reference maps between $\mathbb{R}^m$ and $\mathbb{R}^n$ and compute partial derivatives locally. Let's go over how to do this.



The figure above demonstrates the general interaction between a chart (U, φ) on $\mathcal{M}$, the image set $F(U)$, and some nearby charts (V_i, ψ_i) on $\mathcal{N}$. We have the following choices (for each i):

$$\psi_i \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V_i)) \to \psi(F(U) \cap V_i).$$

In the event that the image is entirely contained in one chart on $\mathcal{N}$ this simplifies to:

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(F(U)).$$

One can arrange for the desired containment by taking subcharts $(U' \subseteq U, \varphi|_{U'})$.

• Def: The maps $\psi_i \circ F \circ \varphi^{-1}$ above are called the **coordinate representations of F with respect to (U, φ) and (V_i, ψ_i)**. In the literature, one will ambiguously see one such coordinate representation referred to by "F-hat":

$$\widehat{F} = \psi_i \circ F \circ \varphi^{-1}.$$

• Def: (p.644 [7]) A real map $\boldsymbol{G} : \mathbb{R}^m \to \mathbb{R}^n$ is **smooth** if all of its component functions are infinitely partially-differentiable. In other words, any multi-indexed partial derivative of any component function, $\partial_I G^j$, exists and is continuous.

• Def: (p.35 [7]) Let $\boldsymbol{F} : \boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{N}}$ be a continuous mapping of manifolds. If there exist smooth atlases $\boldsymbol{\mathcal{A}} = \left\{(U_\alpha, \varphi_\alpha)\right\}_{\alpha \in A}$ and $\boldsymbol{\mathcal{B}} = \left\{(V_\beta, \psi_\beta)\right\}_{\beta \in B}$ for $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{N}}$ respectively such that for each pair (α, β), the coordinate representation $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is smooth as a real mapping, then we say $\boldsymbol{F}$ is a **smooth mapping of manifolds**.

Note: In practice, one may be given atlases to work with that need adjusting (via choosing a particular V_i as on the previous page, together with taking subcharts of the domain, etc.) in order for the smoothness to be proven for a particular $\boldsymbol{F}$.

• Def: Considering $\mathbb{R}$ as a smooth manifold with the single global chart $(\mathbb{R}, Id_{\mathbb{R}})$, we have a special case of mappings of manifolds, called **functions on manifolds**, $f : \boldsymbol{\mathcal{M}} \to \mathbb{R}$. The smoothness of which is characterized by existence of a smooth atlas for $\boldsymbol{\mathcal{M}}$ yielding each:

$$Id_{\mathbb{R}} \circ f \circ \varphi_\alpha^{-1} = f \circ \varphi_\alpha^{-1}$$

as being smooth real maps.

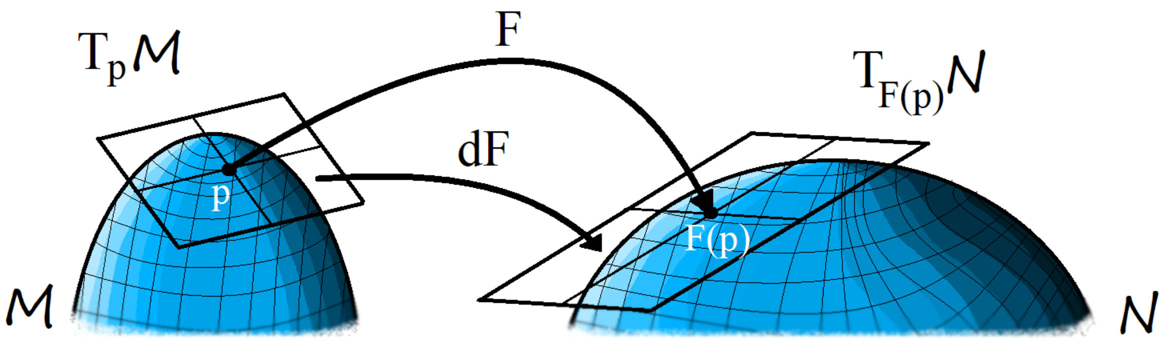
• Def: Some other important special cases being for the tensor fields: $\Phi : \boldsymbol{\mathcal{M}} \to \mathcal{T}_l^k(TM)$. With the manifold structure imposed on the tensor bundles (again visit p.253 [7]), we have a **smooth tensor field** is just a smooth mapping of manifolds with $\boldsymbol{\mathcal{N}} = \mathcal{T}_l^k(T\boldsymbol{\mathcal{M}})$ as defined above.

However, proving smoothness amounts to showing the fields component functions are smooth in every chart (Prop 12.19b p.317 [7]). Recall, with the point dependency now and upper case labelling, tensor fields expanded in a local frame look like:

$$\Phi = \Phi^{i_1 \ldots i_k}_{j_1 \ldots j_l}(p) \cdot \partial_{i_1}(p) \otimes \ldots \otimes \partial_{i_k}(p) \otimes \partial^{j_1}(p) \otimes \ldots \otimes \partial^{j_l}(p)$$

so essentially in all charts, show that $\Phi^{i_1 \ldots i_k}_{j_1 \ldots j_l} \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}$ is a smooth function.

2. Differentials, Pushforwards, and Pullbacks



In this section, we develop maps related to a given smooth mapping of manifolds. Namely: the *differential at a point*, the *global differential*, and the *pushforward* and *pullback maps* which allow us to in essence move vector fields and covector fields (and by extension, tensor fields) from one manifold to another. First, we need to properly define the tangent space at a point.

• Def: (p.54 [7]) Suppose we have a function space $\boldsymbol{C^\infty(\mathcal{M})}$. We define a **derivation at $\boldsymbol{p}$** as a linear map:

$$\boldsymbol{X_p : C^\infty(\mathcal{M}) \to \mathbb{R}}$$

which satisfies the following "product rule" forall $\boldsymbol{f, g \in C^\infty(\mathcal{M})}$:

$$\boldsymbol{X_p(fg) = f(p) \cdot X_p(g) + g(p) \cdot X_p(f)}.$$

• Def: We call the vector space of all derivations at $\boldsymbol{p}$, the **tangent space at $\boldsymbol{p}$**, and denote it by $\boldsymbol{T_p\mathcal{M}}$. We also call the elements of this space **tangent vectors at $\boldsymbol{p}$**.

Notes: These are meant to emulate directional derivative operators in $\mathbb{R}^{\boldsymbol{n}}$. One can now proceed to define vector fields etc. using this definition (see Section I.3 again).

Without further ado...

• Def: (p.55 [7]) The **differential at $\boldsymbol{p}$**, of a smooth mapping of manifolds $\boldsymbol{F : \mathcal{M} \to \mathcal{N}}$, is defined by:

$$\boldsymbol{dF_p : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}}$$
$$\boldsymbol{X_p \mapsto dF_p(X_p)}$$

where the derivation $\boldsymbol{dF_p(X_p)}$ at $\boldsymbol{F(p)}$ is defined to act on $\boldsymbol{f \in C^\infty(\mathcal{N})}$ by:

$$\boldsymbol{[dF_p(X_p)](f) := X_p(f \circ F)}$$

[Exercise: Show $\boldsymbol{dF_p(X_p)}$ satisfies $\boldsymbol{C^\infty(\mathcal{N})}$-linearity and the product rule.]

Let's look at the **coordinate representation of the differential (in bases)** now. Quick note, we identify $T_p\mathbb{R}^n \cong \mathbb{R}^n$, however, we keep track of the point in the notation. So the same basis can be used for both the coordinate system and the tangent space in a chart (up to the point) as we will see.

Given two charts $(U, \varphi)_p$ and $(V, \psi)_{F(p)}$ with $F(U) \subseteq V$, the *coordinate representation for* F is:

$$\widehat{F} := \psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(F(U))$$

for which the *differential of* $\hat{F}$ *at* $\varphi(p)$ is given by:

$$d\hat{F}_{\varphi(p)} : T_{\varphi(p)}\big(\varphi(U)\big) \to T_{\psi(F(p))}\big(\psi(F(U))\big).$$

Then, given the bases $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}\}$ and $\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\}$ for the respective coordinate systems $\varphi(U)$ and $\psi(V)$, we have for $f \in C^\infty(\psi(F(U)))$:

$$\begin{aligned}\left[d\hat{F}_{\varphi(p)}\left(\tfrac{\partial}{\partial x^j}\Big|_{\varphi(p)}\right)\right](f) &:= \tfrac{\partial}{\partial x^i}\Big|_{\varphi(p)}\big(f \circ \widehat{F}\big) \\ &= \sum_{i=1}^{n} \tfrac{\partial}{\partial y^i}\big(f \circ \widehat{F}(\varphi(p))\big) \cdot \tfrac{\partial}{\partial x^j}\widehat{F}^i(\varphi(p)) && \text{[via Chain Rule (p.647 [7])]} \\ &= \tfrac{\partial \widehat{F}^i}{\partial x^j}\Big|_{\varphi(p)} \cdot \tfrac{\partial}{\partial y^i}\Big|_{\widehat{F}(\varphi(p))}(f) && \text{[rewriting]}\end{aligned}$$

so that abstracting from f, we get:

$$d\hat{F}_{\varphi(p)}\left(\frac{\partial}{\partial x^j}\bigg|_{\varphi(p)}\right) = \frac{\partial \widehat{F}^i}{\partial x^j}\bigg|_{\varphi(p)} \cdot \frac{\partial}{\partial y^i}\bigg|_{\widehat{F}(\varphi(p))}.$$

So we conclude that the matrix representation for $d\hat{F}_{\varphi(p)}$ in the two stated bases is exactly the usual jacobian matrix $\left(\frac{\partial \widehat{F}^i}{\partial x^j}\right)_{ij}$ from calculus. Finally, linearity of matrices gives:

$$[d\hat{F}_{\varphi(p)}]^{\beta_2}_{\beta_1}[\hat{X}_{\varphi(p)}]_{\beta_1} = \left[\hat{X}^k \frac{\partial \widehat{F}^1}{\partial x^k}, \ \ldots \ , \hat{X}^k \frac{\partial \widehat{F}^n}{\partial x^k}\right]^t_{\beta_2}\Bigg|_{\varphi(p)}. \qquad \textbf{(II.2.A)}$$

where the sum for k ranges from $\{1, \ldots, m\}$ in each component; Also the t is for transpose since vectors are "column vectors" and covectors are "row vectors".

(Continues)

Now for the related maps...

• Def: We define the **global differential** of a smooth map by

$$dF : T\mathcal{M} \to T\mathcal{N}$$
$$dF\Big((p, X_p)\Big) := \Big(F(p),\ dF_p(X_p)\Big),$$

using $dF_p(X_p)$ from before. Summarizing, this says: $dF(X)(f) = X(f \circ F)$

• Def: (p.183 [7]) If a smooth map $F : \mathcal{M} \to \mathcal{N}$ is also invertible, then we may define the so called **pushforward** of vector fields, by defining the image section:

$$F_* : \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{N})$$
$$[F_*(X)](q) := \Big(q,\ dF_{F^{-1}(q)}(X_{F^{-1}(q)})\Big),$$

where $q \in \mathcal{N}$ (recall we want a map $\mathcal{N} \to T\mathcal{N}$).

Note: We need the inverse to be smooth as well if we want the resulting field in the image to be smooth (see Prop 8.19 (p.183) [7]). Such bi-C^∞ maps are called **diffeomorphisms**.

• Def: (p.284-285 [7]) For a smooth map (not necessarily diffeo), define the **pullback** of covector fields:

$$F^* : \Gamma(T^*\mathcal{N}) \to \Gamma(T^*\mathcal{M})$$
$$[F^*(\omega)](p) := (p, \widetilde{\omega}_p)$$

Where for $X_p \in T_p\mathcal{M}$, $\qquad \widetilde{\omega}_p(X_p) := \omega_{F(p)}(dF_p(X_p)).$

Note: This last line is a point-wise definition of pullback for covectors (similar to the point-wise differential). We just apply the differential at a point to derivations at a point and then apply the original covector. More succinctly for fields we write:

$$F^*\omega(X) = \omega(dF(X))$$

[Exercise: Use equation II.2.A on the previous page to create a coordinate/basis representation of the pullback.]

[Exercise: For a diffeomorphism, F, we can define a generalized pushforward/pullback map:

$$\Gamma(\mathcal{T}^k_l(T\mathcal{M})) \leftrightarrow \Gamma(\mathcal{T}^k_l(T\mathcal{N}))$$

using the two we defined above as base cases. It gets ugly, but I've done it, try writing it out. Just pushforward vector factors of the tensors and pullback covector factors! Start by supposing

$$\Phi = \Phi^{i_1 \ldots i_k}_{j_1 \ldots j_l} \partial_{i_1} \otimes \ldots \otimes \partial_{i_k} \otimes \partial^{j_1} \otimes \ldots \otimes \partial^{j_l} \in \Gamma(\mathcal{T}^k_l(U)),$$

then define $F_*(\Phi)$ and $F^*(\Phi)$ via...]

3. Contraction, Outer/Inner Products, Metrics, and Index Raising/Lowering

This section was motivated by Kay's text (p.43-44 and p.55 of [6]), but adapted to our notations and placed in conjunction with Lee [7]. Note that this section has a few nonstandard notations for the sake of clarity.

• Def: For $k, l \geq 1$, we define a map called **contraction**:

$$C : \Gamma(\mathcal{T}^k_l(T\mathcal{M})) \times \{1, ..., k\} \times \{1, ..., l\} \rightarrow \Gamma(\mathcal{T}^{k-1}_{l-1}(T\mathcal{M}))$$

$$C\Big(\Phi, r, s\Big)^{i_1 \dots \widehat{i_r} \dots i_k}_{j_1 \dots \widehat{j_s} \dots j_l} := \Phi^{i_1 \dots A \dots i_k}_{j_1 \dots A \dots j_l}$$

That is, the components of the contracted tensor are the original tensor's, summed over the identified indices. The hat character denotes removal of the indices.

Example: Take $\Phi = \Phi^i_j \partial_i \otimes \partial^j$. Then $C(\Phi, 1, 1) = \Phi^A_A = \Phi^1_1 + ... + \Phi^n_n$.

• Def: The **outer product** is just the tensor product as we have already defined:

$$OP : \Gamma(\mathcal{T}^k_l(T\mathcal{M})) \times \Gamma(\mathcal{T}^m_n(T\mathcal{M})) \rightarrow \Gamma(\mathcal{T}^{k+m}_{l+n}(T\mathcal{M})),$$
$$OP(\Phi, \Psi) := \Phi \otimes \Psi.$$

The components in a local frame are then:

$$OP(\Phi, \Psi)^{i_1 \dots i_k i_{k+1} \dots i_{k+m}}_{j_1 \dots j_l j_{l+1} \dots j_{l+n}} := \Phi^{i_1 \dots i_k}_{j_1 \dots j_l} \cdot \Psi^{i_{k+1} \dots i_{k+m}}_{j_{l+1} \dots j_{l+n}}$$

• Def: Given $\Phi \in \Gamma(\mathcal{T}^k_l(T\mathcal{M}))$ and $\Psi \in \Gamma(\mathcal{T}^m_n(T\mathcal{M}))$, the **inner product** of Φ and Ψ over a co-variant index, s, from Φ and a contra-variant index, r, from Ψ is given by:

$$IP : \Gamma(\mathcal{T}^k_l(T\mathcal{M})) \times \{1, ..., l\} \times \Gamma(\mathcal{T}^m_n(T\mathcal{M})) \times \{1, ..., m\} \rightarrow \Gamma(\mathcal{T}^{k+m-1}_{l+n-1}(T\mathcal{M}))$$

$$IP(\Phi, s, \Psi, r) := C\Big(OP(\Phi, \Psi), k + r, s\Big)$$

The components in a local frame are thus:

$$IP\Big(\Phi, s, \Psi, r\Big)^{i_1 \dots i_k i_{k+1} \dots \widehat{i_{k+r}} \dots i_{k+m}}_{j_1 \dots \widehat{j_s} \dots j_l j_{l+1} \dots j_{l+n}} := \Phi^{i_1 \dots i_k}_{j_1 \dots A \dots j_l} \cdot \Psi^{i_{k+1} \dots A \dots i_{k+m}}_{j_{l+1} \dots j_{l+n}}$$

Example: Let $\omega = \omega_i \partial^i$ and $X = X^j \partial_j$. Then $\omega \otimes X = \omega_i X^j \partial^i \otimes \partial_j$ and hence contracting across the only two indices available yields the inner product: $\omega_1 X^1 + ... + \omega_n X^n$.

Note that the blue symbology is the one to pay attention to, the rest is for intuition/coding purposes and is nonstandard. We can summarize by saying *inner product is a contraction of the outer product*. One gets a similar definition instead contracting contra- to co- which we will use soon much later.

Riemannian and Psuedo-Riemannian Metrics

• <u>Def:</u> (p.327-328 [7]) Given a smooth manifold $\boldsymbol{\mathcal{M}}$, we define a **Riemannian metric** to be a tensor field $\boldsymbol{g} \in \boldsymbol{\Gamma}(\boldsymbol{\mathcal{T}_2^0}(\boldsymbol{T\mathcal{M}}))$, that is smooth, *symmetric*, and *positive-definite* at each point.

Symmetric means $\forall \boldsymbol{i}, \boldsymbol{j} \in \{\boldsymbol{1}, \ldots, \boldsymbol{n}\}$ we have $\boldsymbol{g_{ij}} = \boldsymbol{g_{ji}}$ in any frame. And by **positive-definite** at each point, we have:

$$\forall \boldsymbol{X_p} \in \boldsymbol{T_p\mathcal{M}}, \qquad \boldsymbol{g_p}(\boldsymbol{X_p}, \boldsymbol{X_p}) \geq \boldsymbol{0} \text{ with equality iff } \boldsymbol{X_p} = \boldsymbol{0}$$

• <u>Def:</u> (p.328 [7]) A **Riemannian manifold** is a pair $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$, where $\boldsymbol{\mathcal{M}}$ is a smooth manifold and $\boldsymbol{g}$ is a choice of Riemannian metric on $\boldsymbol{\mathcal{M}}$.

See (Prop. 13.3 (p.329) [7]) for "Existence of Riemannian Metrics" proof, using *partitions of unity*.

<u>Notes:</u> (p.329-337) Riemannian metrics give a traditional vector space inner product for each tangent space $\boldsymbol{T_p\mathcal{M}}$, so we use the notation:

$$\boldsymbol{g_p}(\boldsymbol{X_p}, \boldsymbol{Y_p}) = \langle \boldsymbol{X_p}, \boldsymbol{Y_p} \rangle_{\boldsymbol{g}}.$$

From this, we may define the **norm**, $|\boldsymbol{X_p}|_{\boldsymbol{g}} := \sqrt{\boldsymbol{g_p}(\boldsymbol{X_p}, \boldsymbol{X_p})}$ of a tangent vector at a point and also the **angle** between two tangent vectors in the usual way: $\boldsymbol{cos\theta} = \frac{\langle \boldsymbol{X_p}, \boldsymbol{Y_p} \rangle}{|\boldsymbol{X_p}| \cdot |\boldsymbol{Y_p}|}$. This gives rise to **orthogonality** and **orthonormality**, which then extends to frames and also gives decompositions of tangent spaces of *Riemannian submanifolds* into "tangent" and "normal" directions etc.

Notice that this "metric" is defined over $\boldsymbol{T\mathcal{M}}$ and not over $\boldsymbol{\mathcal{M}}$ (that is, we don't measure distance between points on the manifold). However it can be used to define a traditional (topological) metric on the manifold itself [See (p.93-94 [8])].

Alternatively, for physics applications we have:

• <u>Def:</u> (p.343-344 [7]) A **psuedo-Riemannian metric** on a smooth manifold $\boldsymbol{\mathcal{M}}$ is a smooth, symmetric 2-tensor field (as above) except instead of positive definite at each point, we require *non-degeneracy* at each point and also we require this tensor field to have the same *signature* everywhere on $\boldsymbol{\mathcal{M}}$. The matrix of $\boldsymbol{g}$ is **non-degenerate** at $\boldsymbol{p} \in \boldsymbol{\mathcal{M}}$ if

$$\forall \boldsymbol{X_p} \neq \boldsymbol{0} \in \boldsymbol{T_p\mathcal{M}}, \ \exists \boldsymbol{Y_p} \in \boldsymbol{T_p\mathcal{M}}, \text{ such that } \boldsymbol{g_p}(\boldsymbol{X_p}, \boldsymbol{Y_p}) \neq \boldsymbol{0}.$$

The **signature of** $\boldsymbol{g}$ is defined to be the quantity $(\boldsymbol{P}, \boldsymbol{N})$, where $\boldsymbol{P}$ and $\boldsymbol{N}$ are (respectively) the number of *positive* and *negative* eigenvalues in the matrix for $\boldsymbol{g}$.

<u>Note:</u> (p.344 [7]) Not every manifold admits psuedo-Riemannian metrics.

Raising and Lowering Indices of Tensors

For the following, assume we have either a smooth Riemannian or psuedo-Riemannian manifold (M, g) and a given local frame $\{\partial_1, \ldots, \partial_n\}$ for the tangent bundle. Both positive definite and non-degenerate matrices can be shown to be invertible [Exercise: Prove this], so g has an invertible component matrix.

• Def: Define the two tensor fields (sometimes referred to as the **metric** and **conjugate metric**):

$$g = g_{ij}\partial^i \otimes \partial^j \qquad \text{and} \qquad h := g^{ij}\partial_i \otimes \partial_j$$

such that $g^{ik}g_{kj} = \delta^i_j = \delta^j_i = g_{ik}g^{kj}$. That is, the matrix for h is the inverse for that of g.

We can use the metric and its conjugate to convert tensors to different valences. We do this through the *inner product* defined at the beginning of this section (recall the blue symbology).

• Def: Let $\Phi = \Phi^{i_1\ldots i_k}_{j_1\ldots j_l}\partial_{i_1} \otimes \ldots \otimes \partial^{j_l}$ and suppose we wish to raise the j_s index or lower the i_r index, respectively yielding valences $(k+1, l-1)$ or $(k-1, l+1)$. Then, we have the following inner products listed in components:

$$IP\big(\Phi, s, h, 1\big)^{i_1\ldots i_k i_{k+1}}_{j_1\ldots\widehat{j_s}\ldots j_l} := \Phi^{i_1\ldots i_k}_{j_1\ldots A\ldots j_l} \cdot g^{Ai_{k+1}}$$

$$IP\big(g, 1, \Phi, r\big)^{i_1\ldots\widehat{i_r}\ldots i_k}_{j_0 j_1\ldots j_l} := g_{Aj_0} \cdot \Phi^{i_1\ldots A\ldots i_k}_{j_1\ldots j_l}$$

Notes: This is phrased for coding purposes, but the message should be clear in **blue**. The index we wish to affect gets contracted with an opposing index in either the metric or its conjugate and what's left is an additional opposing index. To be pedantic, one can reindex the i_{k+1} or j_0 to be the same as the affected index (so that we really did raise or lower it). Lastly, note that this process can be repeated to make tensors fully co- or contra-variant!

Example: Let $X = X^i\partial_i$. We can turn this to a covector field by lowering the component index with the metric! After reindexing and using symmetry of g, we write:

$$X_i := g_{ij}X^j$$

With the new covector field $X_i\partial^i := g_{ij}X^j\partial^i$, we can raise the index back with the conjugate (reindexing when necessary):

$$g^{ij}X_j := g^{ij}(g_{jk}X^k) = \delta^i_k X^k = X^i$$

For this reason, for raising covector components, we also write:

$$\omega^i := g^{ij}\omega_j.$$

<< Chapter III : The Calculus >>

In this chapter we develop the notions of *integration* and *derivation* of certain tensor fields. The natural starting point is a discussion on symmetric tensors. Continuing the discussion on manipulation, we also will see how to *symmetrize* or *skew-symmetrize* tensor fields or products thereof.

1. Symmetric and Skew-Symmetric Tensors (esp. Differential Forms)

Briefly, a *symmetry* is an operation applied to an object that leaves the state of the object invariant. In our case, we look at *permutations* as the "operations" acting on *tensor indices/arguments* as the "objects" and the "state" is just the *value* of the field's component with the affected index.

Coincidentally, permutations come from groups called the *Symmetric Groups* $\boldsymbol{S_k}$. After defining symmetric tensor fields, skew-symmetric tensor fields are analogously defined and restricting to purely covariant skew-symmetric tensor fields gives us the differential forms we are after for integration. Symmetric tensor fields will show up in the next chapter.

This section takes ideas from (Ch.2.15 [2]) and (Ch.12 and 14 [7]) as well as uses [3] for algebra reference. The subsection directory is given below.

1.1 On Symmetric Tensors

• <u>Def:</u> For a given valence (k, l), with $k \geq l$, define two *linear group actions*

$$S_k \times \Gamma(\mathcal{T}^k_l(T\mathcal{M})) \to \Gamma(\mathcal{T}^k_l(T\mathcal{M}))$$

of S_k on the set $\Gamma(\mathcal{T}^k_l(T\mathcal{M}))$ by the following:

$${}_{\sigma}(\Phi)^{i_1 \dots i_k}_{j_1 \dots j_l} := \Phi^{i_1 \dots i_k}_{\sigma(j_1) \dots \sigma(j_l)}$$

and

$${}^{\sigma}(\Phi)^{i_1 \dots i_k}_{j_1 \dots j_l} := \Phi^{\sigma(i_1) \dots \sigma(i_k)}_{j_1 \dots j_l}$$

for each $\sigma \in S_k$ and $\Phi \in \Gamma(\mathcal{T}^k_l(T\mathcal{M}))$. The collection of such component functions ranging over all indices gives the tensor field defined in the image. This can be written alternatively as:

$${}_{\sigma}(\Phi)(\partial^{i_1}, \dots, \partial^{i_k}, \partial_{j_1}, \dots, \partial_{j_l}) := \Phi(\partial^{i_1}, \dots, \partial^{i_k}, \partial_{\sigma(j_1)}, \dots, \partial_{\sigma(j_l)})$$

and

$${}^{\sigma}(\Phi)(\partial^{i_1}, \dots, \partial^{i_k}, \partial_{j_1}, \dots, \partial_{j_l}) := \Phi(\partial^{\sigma(i_1)}, \dots, \partial^{\sigma(i_k)}, \partial_{j_1}, \dots, \partial_{j_l}).$$

so that *permuting component indices is equivalent to permuting variables*. For obvious reasons, we can't swap co- and contravariant variables (hence the need for two separate actions).

• <u>Def:</u> Given $\sigma \in S_k$ and Φ as before, we say **σ is a symmetry of Φ** if either:

$${}_{\sigma}(\Phi) = \Phi \qquad \text{or} \qquad {}^{\sigma}(\Phi) = \Phi.$$

We can specify and say a *co-* or *contra-variable symmetry*.

<u>Example:</u> A tensor field being symmetric in the first and third (contra-) variables means $\sigma = (1\ 3)$ is a symmetry of Φ or:

$$\Phi^{i_3 i_2 i_1} = \Phi^{i_1 i_2 i_3}$$

for all values of i_1, i_2, and i_3.

• <u>Def:</u> We say Φ is **totally symmetric** if it is invariant under the action of all $\sigma \in S_k$ (both up and down). The **set of all symmetric (k, l)-tensor fields** is given a special symbol $\Gamma(\Sigma^k_l(T\mathcal{M}))$. The **symmetrization map**:

$$Sym : \Gamma(\mathcal{T}^k_l(T\mathcal{M})) \to \Gamma(\Sigma^k_l(T\mathcal{M}))$$

$$Sym(\Phi) := \frac{1}{k!}\left(\sum_{\sigma \in S_k} {}^{\sigma}(\Phi) +_{\sigma} (\Phi) \right)$$

gives $\bigoplus_{k,l \in \mathbb{N}} \Gamma(\Sigma^k_l(T\mathcal{M}))$ a *graded algebra* structure with the bilinear operator, **symmetric product**, given by:

$$\Phi\Psi := Sym(\Phi \otimes \Psi).$$

[Exercise: Prove this algebraic statement.]

1.2 On Skew-Symmetric Tensors

• Def: (p.315 [7]) A $(\boldsymbol{k},\boldsymbol{l})$-tensor field $\boldsymbol{\Phi}$, with $(\boldsymbol{k} \geq \boldsymbol{l})$, is called **totally skew-symmetric** or **alternating** if for all transpositions $\boldsymbol{\sigma} \in \boldsymbol{S_k}$, we have:

$$_{\sigma}(\boldsymbol{\Phi}) = -\boldsymbol{\Phi} \qquad \text{and} \qquad ^{\sigma}(\boldsymbol{\Phi}) = -\boldsymbol{\Phi}$$

respectively. Of course, we only mention co- or contra- if it exhibits one or the other but not both for all $\boldsymbol{\sigma}$. Another way to state the above is that alternating tensors have negatives pop out when you swap any two variables.

• Def: (p.351-358 [7])
Analogous to before with $\boldsymbol{k} \geq \boldsymbol{l}$, the **set of all alternating $(\boldsymbol{k},\boldsymbol{l})$-tensor fields** is given a special symbol: $\boldsymbol{\Gamma(\Lambda^k_l(T\mathcal{M}))}$. And via the **skew-symmetrization map** (a.k.a. **alternator map**):

$$\boldsymbol{Alt} : \Gamma(\mathcal{T}^k_l(\boldsymbol{T\mathcal{M}})) \to \Gamma(\Lambda^k_l(\boldsymbol{T\mathcal{M}}))$$

$$\boldsymbol{Alt}(\boldsymbol{\Phi}) := \frac{1}{k!}\Bigg(\sum_{\sigma\in S_k}(\boldsymbol{sgn}(\boldsymbol{\sigma}))\cdot\Big(^{\sigma}(\boldsymbol{\Phi}) +_{\sigma}(\boldsymbol{\Phi})\Big)\Bigg),$$

the direct sum $\bigoplus\limits_{k,l\in\mathbb{N}} \boldsymbol{\Gamma(\Lambda^k_l(T\mathcal{M})}$ gets the structure of a *graded algebra* with the bilinear operator, called the **skew-symmetric product**:

$$\boldsymbol{Alt}(\boldsymbol{\Phi} \otimes \boldsymbol{\Psi})$$

Notice the difference here, in the above sum, the terms are weighted by the *sign of the permutation* in order to introduce the alternating nature of the tensors components (on top of the symmetrization).

(Continues)

1.3 On Differential Forms

We now restrict to valence $(k, l) := (0, k)$ alternating tensor fields. These are the objects we will use in integration in the next section! So let's lock down their definitions.

• Def: (p.355-356 [7])
An element of $\Gamma(\Lambda_k^0(T\mathcal{M}))$ is called a **differential form** (or more specifically a **k-form**). Again, these are just purely covariant tensor fields whose components have a negative pop out when you swap two indices.

• Def: The algebra $\bigoplus_{k\in\mathbb{N}} \Gamma(\Lambda_k^0(T\mathcal{M}))$ can be equipped with a scaled variant of the *skew-symmetric product* called the **wedge product**, that is convenient for computation (as we'll see):

$$\Phi \wedge \Psi := \frac{(k_1+k_2)!}{k_1!\cdot k_2!}\cdot Alt(\Phi\otimes\Psi)$$

$$= \frac{1}{k_1!\cdot k_2!}\left(\sum_{\sigma\in S_{k_1+k_2}} (sgn(\sigma))\cdot \ _\sigma(\Phi\otimes\Psi)\right) \qquad (\star)$$

for k_1-form Φ and k_2-form Ψ, the result is a (k_1+k_2)-form.

• Def: (p.352 [7]) We introduce some key players in the mathematics of differential forms, called *elementary k-forms.* We denote a **strictly-ordered multi-index** by $I := (i_1, ..., i_k)$, where $i_1 < ... < i_k$. Then for such an index, using the coordinate frame on a neighborhood, $\{\partial_1, ..., \partial_n\}$, and its dual, define the **elementary k-form**:

$$\partial^I(X_1, ..., X_k) := det\begin{bmatrix}\partial^{i_1}(X_1) & ... & \partial^{i_1}(X_k)\\ \vdots & \ddots & \vdots\\ \partial^{i_k}(X_1) & ... & \partial^{i_k}(X_k)\end{bmatrix}$$

for vector fields X_i.

Notes: Recall from linear algebra, that for fixed k, the determinant is a real-valued function of $k\times k$ matrices that is linear in each column argument and switches sign when any two columns are permuted (sounds like a k-form to me!). The catch is that here, the vector fields X_i we are using as inputs have n components - so there are choices available for ways to crop the matrix $[X_1...X_k]$ to be $k\times k$. These choices give rise to the different elementary k-forms.

Another property we utilize of determinant is that any repeated (or even linearly dependent) row or columns make the result 0. So for nonzero results, our crops should have unique row components and for uniqueness up to sign, we choose the standard ordering of the indices. Hence the strictly-ordered multi-index. There are $\binom{n}{k}$ such multi-indices.

(Continues)

1.4 Wedge Product of Elementary Differential Forms

Suppose $\boldsymbol{I = (i_1, ..., i_k)}$ and $\boldsymbol{J = (i_{k+1}, ..., i_{k+l})}$ are strictly ordered multi-indices such that $\boldsymbol{i_k < i_{k+1}}$ and $\boldsymbol{k + l \leq n}$. Then $\boldsymbol{IJ := (i_1, ..., i_{k+l})}$ is also a strictly ordered multi-index. Thus, we can define $\boldsymbol{\partial^I}$, $\boldsymbol{\partial^J}$, and $\boldsymbol{\partial^{IJ}}$ as elemenatary $\boldsymbol{k}$, $\boldsymbol{l}$, and $\boldsymbol{k + l}$-forms respectively. WTS that $\boldsymbol{\partial^I \wedge \partial^J = \partial^{IJ}}$.

• <u>Proof:</u> (Based off of p.355-356 [7])
We need to show the two tensors have equal components. That is, forall $\boldsymbol{a_b \in \{1, ..., n\}}$,

$$\boldsymbol{(\partial^I \wedge \partial^J)_{a_1 \ldots a_{k+l}} = (\partial^{IJ})_{a_1 \ldots a_{k+l}}},$$

which we obtain by feeding each side the sequences $\boldsymbol{(\partial_{a_1}, ..., \partial_{a_{k+l}})}$. And there are cases for what these sequences could be.

<u>Case 1:</u> The sequence subscripts match that of $\boldsymbol{IJ}$.

Then by $\boldsymbol{(\bigstar)}$ on the previous page, we obtain:

$$\boldsymbol{\partial^I \wedge \partial^J(\partial_I, \partial_J) := \frac{1}{k! \cdot l!}\left(\sum_{\sigma \in S_{k+l}} (sgn\sigma) \cdot \;{}_{\sigma}(\partial^I \otimes \partial^J)\right)(\partial_{i_1}, ..., \partial_{i_k}, \partial_{i_{k+1}}, ..., \partial_{i_{k+l}})}$$

$$\boldsymbol{= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (sgn\sigma) \cdot \partial^I(\partial_{\sigma(i_1)}, ..., \partial_{\sigma(i_k)}) \cdot \partial^J(\partial_{\sigma(i_{k+1})}, ..., \partial_{\sigma(i_{k+l})})}\ .$$

Now, we argue that the only terms in the sum that are nonzero are ones in which $\boldsymbol{\sigma = \alpha\beta}$, for $\boldsymbol{\alpha \in S_k}$ permuting the first $\boldsymbol{k}$ arguments and $\boldsymbol{\beta \in S_l}$ permuting the last $\boldsymbol{l}$ arguments (otherwise we get columns of zeros in the determinant for missing indices). Together with the multiplicative property of $\boldsymbol{sgn}$, we get:

$$\boldsymbol{= \frac{1}{k!l!} \sum_{\substack{\alpha \in S_k \\ \beta \in S_l}} (sgn\alpha)(sgn\beta)\partial^I(\partial_{\alpha(i_1)}, ..., \partial_{\alpha(i_k)}) \cdot \partial^J(\partial_{\beta(i_{k+1})}, ..., \partial_{\beta(i_{k+1})})}$$

$$\boldsymbol{= \left(\frac{1}{k!} \sum_{\alpha \in S_k} (sgn\alpha)\ {}_{\alpha}(\partial^I)(\partial_I)\right) \cdot \left(\frac{1}{l!} \sum_{\beta \in S_l} (sgn\beta)\ {}_{\beta}(\partial^J)(\partial_J)\right)}$$

$$\boldsymbol{= Alt(\partial^I)(\partial_I) \cdot Alt(\partial^J)(\partial_J) = \partial^I(\partial_I) \cdot \partial^J(\partial_J)}$$

by the fact that $\boldsymbol{Alt(\Phi) = \Phi}$ for alternating tensors [Exercise: Prove this]. And the last term in the equality is:

$$\boldsymbol{= 1 = \partial^{IJ}(\partial_I, \partial_J)}.$$

<u>Case 2:</u> Otherwise. Think determinant. Rearrangements of $\boldsymbol{IJ}$ can be reduced to Case 1 canceling out the signs on each side of the equality attained from permuting back to $\boldsymbol{IJ}$. Non-rearrangements of $\boldsymbol{IJ}$ indicate existence of either a repeated index or an index that does not appear in the list $\boldsymbol{IJ}$, in either of these scenarios we get both sides of the equality equal to zero since we'd get respectively: repeated columns in the determinant or a column of zeros. ■

1.5 Properties of Differential Forms

We conclude this section by listing a few important statements, some coming from (**Prop 14.11** on p.356 and **Lemma 14.16** from p.361 [7]).

1.) Bases for $\Lambda_k(T_p\mathcal{M})$:
For each point $p \in \mathcal{M}$, $\left\{\partial^I \mid I \text{ a strictly ordered multi-index}\right\}$ is a basis for $\Lambda_k(T_p\mathcal{M})$.
For the proof, see p.353 [7].

2.) Associativity of Wedge Product:
As a consequence of our proof on the previous page, we get *associativity of wedge product in* $\Lambda_k(T_p\mathcal{M})$ by:

$$(\partial^I \wedge \partial^J) \wedge \partial^K = \partial^{IJ} \wedge \partial^K = \partial^{IJK} = \partial^I \wedge \partial^{JK} = \partial^I \wedge (\partial^J \wedge \partial^K).$$

and using (1) to extend by multi-linearity to arbitrary differential forms.

3.) Decomposition and Alternative Notation:
As a further consequence and induction, we have for a given strictly ordered multi-index, $I = (i_1, ..., i_k)$, the *elementary k-forms* decompose into:

$$\partial^I = \partial^{i_1} \wedge ... \wedge \partial^{i_k}.$$

In a chart (U, φ) with coordinates $\varphi(p) = (x^1(p), ..., x^n(p))$, the coordinate co-vector fields defined point-wise by the pullback, $\partial^i|_p := \varphi^*(\partial^i|_{\varphi(p)})$, are more commonly denoted $dx^i_p := \partial^i_p$, so we can rewrite the elementary k-forms on a neighborhood as:

$$\partial^I = dx^{i_1} \wedge ... \wedge dx^{i_k}.$$

Then for an arbitrary k-form defined on (U, φ), $\Phi \in \Gamma(\Lambda_k(TU))$, we write:

$$\Phi = \Phi_I \partial^I = \Phi_{i_1 ... i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}$$

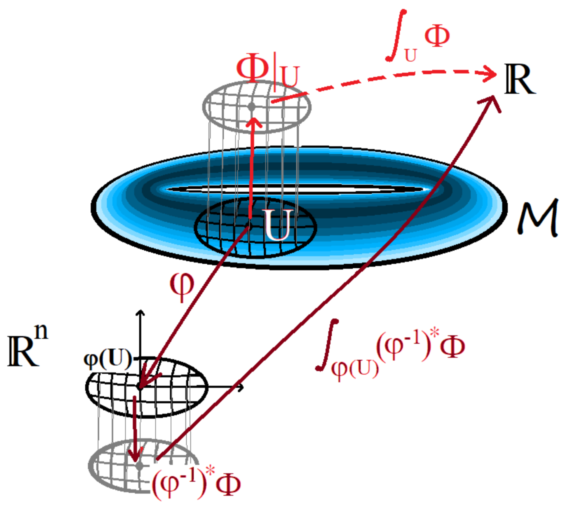
with the understanding that the sum is taken over the proper multi-indices. If this looks like an integrand, that's because it will be soon!

4.) Pullback Formula for Differential Forms:
For a smooth mapping of manifolds $F : \mathcal{M} \to \mathcal{N}$ and $\Phi = \Phi_I dy^{i_1} \wedge ... \wedge dy^{i_k}$ defined in a smooth chart on N, we have:

$$F^*(\Phi) := (\Phi_I \circ F) \cdot d(y^{i_1} \circ F) \wedge ... \wedge d(y^{i_k} \circ F),$$

which follows easily running out the pullback formula for covariant tensor fields applied to vector fields on $\mathcal{M}$, together with the determinant definition of elementary k-forms and finally using the chain rule to absorb the dF's [Exercise: Details!].

2. Integration on Manifolds, Orientations, and Partitions of Unity



(Figure: Integrating n-Forms Over A Chart)

Import Topology, Real Analysis, and Measure Theory for a deeper understanding of this section [see overall section citations here]. Since our focus is on the tensor manipulations, we keep it very superficial in what follows, emulating the *line/surface integrals* from multi-variable calculus.

• Def: (p.404 [7]) Suppose Φ is a continuous n-form on an *oriented* manifold $\mathcal{M}$ that is *compactly supported* in a single chart (U, φ) that is either *positively* OR *negatively oriented*. Then we define the **integral of Φ over U** to be respectively:

$$\int_{\mathcal{M}} \Phi := \pm \int_{\varphi(U)} (\varphi^{-1})^* \Phi.$$

That is, we pullback the differential form into the chart and then erasing the wedges, '$\wedge$', as they appear in the differential form, we can apply our knowledge of integrals in $\mathbb{R}^n$.

• Def: (p.405 [7]) Suppose instead that Φ is a continuous n-form whose *(compact) support* is contained in a finite covering of charts $\{(U_i, \varphi_i)\}_{i=1}^m$ on an *oriented* manifold $\mathcal{M}$ and suppose $\{f_i\}_{i=1}^m$ is a *partition of unity, subordinate to the cover*. Define the **integral of Φ over $\mathcal{M}$** to be:

$$\int_{\mathcal{M}} \Phi := \sum_{i=1}^{m} \left(\int_{\mathcal{M}} f_i \cdot \Phi \right) = \sum_{i=1}^{m} \left((\pm)_{\varphi_i} \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* (f_i \cdot \Phi) \right).$$

Props 16.4 and 16.5 (pgs. 404-405) say these definitions are independent of the chart chosen containing the support and independent of the cover and partition of unity chosen.

(Continues)

We got the main attraction out of the way pretty quickly to get the point across: as long as your differential form satisfies the hypothesis of the definitions, it can be integrated according to the formulas given as well as the pullback formula for differential forms from Section III.1.5. This assumes further that you know the *orientations of the charts* (to get the signs), and that you have a *partition of unity* at your disposal – subordinate to a covering by charts of $\boldsymbol{supp(\Phi)}$.

Which types of functions or forms other than continuous ones can be integrated and over which types of domains is a subject we will not touch – see package import at the beginning of the section. However, we need to go through and at least define the italicized terms above, which we will do next.

One final note, $\boldsymbol{k}$-forms (for $\boldsymbol{k \in \{1, \ldots, n\}}$) can also be integrated, except they will be defined over $\boldsymbol{k}$-dimensional *submanifolds* the way we have here.

Subsections:

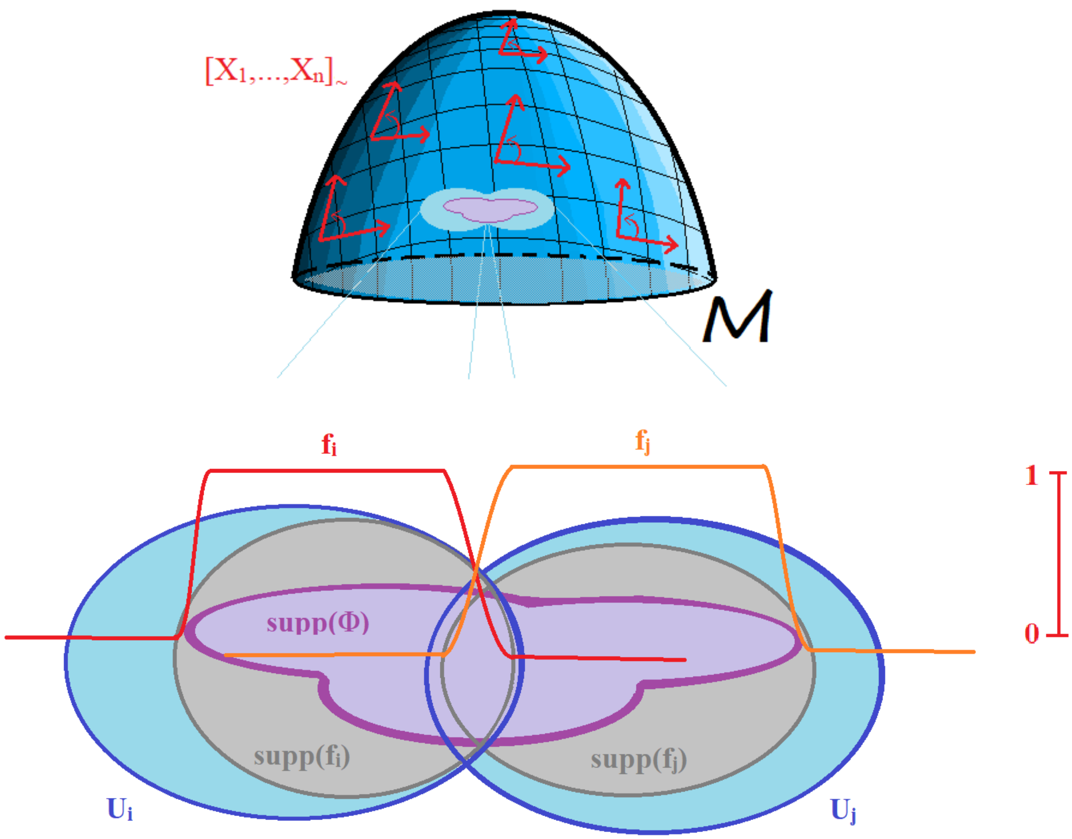


Fig: Visualizing Orientations (top) and Elements of Partitions of Unity As Bump Functions (bottom)

2.1 Orientations

• Def: (p.378 [7]) Given two ordered bases $\boldsymbol{\alpha} = \{v_1, ..., v_n\}$ and $\boldsymbol{\beta} = \{w_1, ..., w_n\}$ for a vector space $\boldsymbol{V}$, we say the two bases are **consistently oriented** if the transition matrix between them has positive determinant.

We may define a relation on the collection of all bases for $\boldsymbol{V}$ (which yields an equivalence relation):

$$\boldsymbol{\alpha} \sim \boldsymbol{\beta} \qquad \leftrightarrow \qquad \boldsymbol{det(A) > 0}, \text{ where } \boldsymbol{A} \text{ is the transition matrix between } \boldsymbol{\alpha} \text{ and } \boldsymbol{\beta}.$$

Given any basis $\boldsymbol{\alpha} = \{v_1, ..., v_n\}$, we denote the two equivalence classes under the *consistently oriented relation* by:

$$[v_1, ..., v_n]_\sim \qquad \text{and} \qquad -[v_1, ..., v_n]_\sim$$

respectively called the **positive** and **negative orientations** on $\boldsymbol{V}$ defined by $\boldsymbol{\alpha}$.

Now apply these definitions to each tangent space, $\boldsymbol{V_p = T_p\mathcal{M}}$, at points along the manifold.

• Def: (p.380 [7]) **Orientations of local or global frames** (for the tangent bundle):

$$[X_1, ..., X_n]_\sim$$

are determined pointwise by $[X_1|_p, ..., X_n|_p]_\sim$, with a special case being the **chart orientation**, determined by the local coordinate frame $[\partial_1, ..., \partial_n]_\sim$ (p.381-382 [7]).

We define an **orientation of $\mathcal{M}$** to be a *continuous* choice of positively oriented frame (as opposed to randomly varying orientations at each point). If such a choice exists, we say $\boldsymbol{\mathcal{M}}$ is **oriented**. Not all manifolds are orientable (Ex: Mobius Strip (p.393 [7])).

★ Note: There are various ways to construct an orientation for $\boldsymbol{\mathcal{M}}$, using differential forms (*orientation forms*), atlases, etc. see the reference (p.380+ [7]).

• Def: (p.383 [7]) A mapping of manifolds $\boldsymbol{F : \mathcal{M} \to \mathcal{N}}$ that is a *local diffeomorphism* is said to be **orientation preserving** if at each point, the differential $\boldsymbol{dF_p}$ takes oriented bases to oriented bases and **orientation reversing** if the differential at each point reverses the orientation.

Note: To check if a local diffeo $\boldsymbol{F}$ is orientation preserving, it suffices to check if $\boldsymbol{det(d\widehat{F}) > 0}$ in any oriented smooth charts (p.383 Exercise 15.13b [7]).

2.2 Support and Partitions of Unity

• Def: (p.43/256 [7]) If $\boldsymbol{\Phi}$ is any $\boldsymbol{k}$-form (even $\mathbf{0}$-form) defined on $\boldsymbol{\mathcal{M}}$, the **support of Φ**, is the closure of the set of points where $\boldsymbol{\Phi}$ is nonzero (nonvanishing or not equal to the zero $\boldsymbol{k}$-covector). That is:

$$supp(\Phi) := \overline{\left\{ p \in \mathcal{M} \mid \Phi(p) \neq 0 \right\}}.$$

We say $\boldsymbol{\Phi}$ is **compactly supported** if this set is compact. In such a case, any cover of the support will have a *finite* subcovering!

• Def: (p.43 [7]) (Specialized) Suppose we have chosen a finite sub-covering, $\mathcal{U} = \{U_i\}_{i=1}^m \supseteq supp(\Phi)$, for the compact support of an $\boldsymbol{n}$-form on a manifold $\boldsymbol{\mathcal{M}}$.

We may define a **partition of unity**, **subordinate to $\mathcal{U}$**, as an indexed family of *continuous* functions, $\{f_i : \mathcal{M} \to \mathbb{R}\}_{i=1}^m$, such that:

1.) $\forall i \in \{1, .., m\}, \quad supp(f_i) \subseteq U_i$,

2.) $\forall i \in \{1, .., m\}, \forall p \in \mathcal{M}, \quad 0 \leq f_i(p) \leq 1$, and

3.) $\forall p \in \mathcal{M}, \quad \sum_{i=1}^{m} f_i(p) = 1$.

Notes:
a.) These three axioms give us at each point $p \in \mathcal{M}$ that:

$$\Phi(p) = \left(\sum_{i=1}^{m} f_i(p) \right) \Phi(p) = \sum_{i=1}^{m} (f_i \cdot \Phi)(p),$$

so that the differential form can be split up into a sum of differential forms:

$$f_i \cdot \Phi$$

where for each $\boldsymbol{i}$, the $supp(f_i \cdot \Phi) \subseteq supp(f_i) \subseteq U_i$ and hence can be integrated over U_i with neglible integral on the rest of $\boldsymbol{\mathcal{M}}$. In conclusion, with this tool we can effectively split integrals over multiple charts into a sum of smaller ones. Assuming partitions of unity exist (see Theorem 2.23 (p.43 [7]) for a constructive proof).

b.) We've used partitions of unity to split objects into smaller ones here. They can also be used to glue together smaller ones making bigger objects [Exercise: Read Lemma 2.26 (p.45 [7]) called the Extension Lemma for Smooth Functions. It uses a more general form of partitions of unity given on p.43, but is very worthwhile to see!].

This theory extends to *manifolds with boundary*, *manifolds with corners*, *non-orientable manifolds*, and even in certain cases *non-compactly supported* $\boldsymbol{k}$-forms – see literature.

3. Exterior Derivatives, Stoke's Theorem, and Cohomology

In this section we aim to define a linear operator on differential forms that generalizes the *differential of a function* and encodes some necessary properties to talk about the relationship between *closed* and *exact* forms, *De Rham Cohomology*, and *Stoke's Theorem*. Each of these notions will be defined as we get to them. We assume throughout that everything in sight is smooth (i.e. all partials exist etc.)

• Def: Recall the **differential of a function** $(f : \mathbb{R}^n \to \mathbb{R})$ is given by (in sum convention):

$$df := \partial_i f \partial^i$$

Notice that d thus sends a $\mathbf{0}$-form to a $\mathbf{1}$-form (note that both can trivially be considered alternating tensor fields since there aren't enough indices to permute).

• Def: (p.363 [7]) We define the **k^{th} exterior derivative operators on $\mathbb{R}^n$** by:

$$d_k : \Gamma(\Lambda_k(T\mathbb{R}^n)) \to \Gamma(\Lambda_{k+1}(T\mathbb{R}^n))$$

$$d_k\Phi := (d\Phi_I) \wedge \partial^I$$

which when written out ignoring the strictly ordered multi-index looks like:

$$(d_k\Phi)_{i_1 \dots i_k i_{k+1}} := (\partial_{i_{k+1}} \Phi_{i_1 \dots i_k} \partial^{i_{k+1}}) \wedge \partial^{i_1 \dots i_k}.$$

If we want to adhere to the multi-index convention, some care has to be taken after the fact to incorporate the new index such that $i_k < i_{k+1}$.

Example: (Switching to traditional notation again $\partial^i = dx^i, \Phi = \omega$)

$$d_1(\omega_j dx^j) = (d\omega_j) \wedge dx^j = (\partial_i \omega_j dx^i) \wedge dx^j = (\partial_i \omega_j) dx^i \wedge dx^j$$

$$= \sum_{i<j} (\partial_i \omega_j dx^i \wedge dx^j) + 0 + \sum_{i>j} (\partial_i \omega_j dx^i \wedge dx^j) \qquad [dx^i \wedge dx^i = 0]$$

$$= \sum_{i<j} (\partial_i \omega_j dx^i \wedge dx^j) - \sum_{j<i} (\partial_i \omega_j dx^j \wedge dx^i) \qquad [dx^i \wedge dx^j = -dx^j \wedge dx^i]$$

$$= \sum_{i<j} (\partial_i \omega_j dx^i \wedge dx^j) - \sum_{i<j} (\partial_j \omega_i dx^i \wedge dx^j) \qquad [\text{reindexing}]$$

$$= \sum_{i<j} (\partial_i \omega_j - \partial_j \omega_i) dx^{ij}.$$

(Continues)

• Prop: (Properties of the Exterior Derivative on $\mathbb{R}^n$, p.364 [7]):

0.) $\boldsymbol{d_0 f = df}$ [Base Case]

1.) $\boldsymbol{d_k}$ is linear over $\mathbb{R}$. [$\mathbb{R}$-Linearity]

2.) $\forall\ \boldsymbol{\Phi}$, $\boldsymbol{\Psi}$ ($\boldsymbol{k}$ and $\boldsymbol{l}$-forms resp.), we have: ["Product" Rule]

$$\boldsymbol{d_{k+l}(\Phi \wedge \Psi) = d_k\Phi \wedge \Psi + (-1)^k \Phi \wedge d_l \Psi}$$

3.) $\boldsymbol{d_k \circ d_{k-1} = 0}$ [$d^2 = 0$]

4.) $\boldsymbol{F^*(d_k\Phi) = d_k(F^*\Phi)}$ [Commutes with Pullbacks by Real Maps]

Proof: [Exercise: Hint: Prove first with simple differential forms and invoke linearity. Also, use the pullback formula for differential forms at the end of Section 1 for (4).]

We wish to upgrade this to the entire manifold using *partitions of unity* to patch things together!

★ Def: (p.365-367) We define the **$\boldsymbol{k^{th}}$ exterior derivative operators on $\boldsymbol{\mathcal{M}}$** using the definitions in each chart:

$$\boldsymbol{d_k : \Gamma(\Lambda_k(T\mathcal{M})) \to \Gamma(\Lambda_{k+1}(T\mathcal{M}))}$$

$$\boldsymbol{d_k := \sum_{\alpha \in A} f_\alpha \cdot \left(\varphi_\alpha^* \circ \tilde{d}_k \circ (\varphi_\alpha^{-1})^* \right)}$$

where on the right hand side we have used a partition of unity (in the general sense (p.43 [7]), $\boldsymbol{\{f_\alpha\}_{\alpha \in A}}$, subordinate to the atlas $\boldsymbol{\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}}$ as a cover for $\boldsymbol{\mathcal{M}}$.

In other words, we pullback the $\boldsymbol{k}$-form into a chart, apply the old $\boldsymbol{d_k}$ (tilde added) in $\mathbb{R}^{\boldsymbol{n}}$, and send the result back via the pullback the other way (some would say *conjugate* by $\boldsymbol{\varphi_\alpha^*}$) – patching these results together with the partition of unity gives us what we want.

Notes: See Thm 14.24 and 14.26 (p.365-367) [7] for well-definedness and uniqueness proof of the above as well as the extension of (4) in the real definition to arbitrary mappings of manifolds.

We just defined a more or less explicit, coordinate definition of the exterior derivative operators as they apply to global $\boldsymbol{k}$-forms. There is a *coordinate-invariant* formula (Prop 14.32 p.370 [7]) defined in terms of a *Lie Bracket*, but since we are not covering *Lie Theory*, we leave that to the interested reader.

Stoke's Theorem and De Rham Cohomology

• Def: (p.25 [7]) Define $\mathbb{H}^n := \{x \in \mathbb{R}^n \mid x^n \geq 0\}$. Then a **boundary chart** for a manifold is a chart $(U, \varphi : U \xrightarrow{\cong} \mathbb{H}^n)$ and the **boundary of $\mathcal{M}$**, is given by:

$$\partial\mathcal{M} := \bigcup_{\text{bdry charts}} \varphi^{-1}(\partial\mathbb{H}^n),$$

where of course $\partial\mathbb{H}^n := \{x \in \mathbb{R}^n \mid x^n = 0\}$.

• Thm 16.11 (Stoke's Theorem) (p.411 [7]) Let $\mathcal{M}$ be an oriented smooth n-manifold with boundary (having induced orientation), and let Φ be a compactly supported smooth $(n-1)$-form on $\mathcal{M}$. Then:

$$\int_{\mathcal{M}} d_{n-1}\Phi = \int_{\partial\mathcal{M}} \iota^*_{\partial\mathcal{M}}(\Phi)$$

where $\iota_{\partial\mathcal{M}} : \partial\mathcal{M} \to \mathcal{M}$ is the inclusion map.

Proof: [Exercise: Read [7]. It breaks down into cases, where $\mathcal{M} = \mathbb{H}^n$, $\mathbb{R}^n$,then compact support contained in a single chart, then multiple charts. The bulk of the work is done in the first case and amounts to running out the definitions and showing both sides are equal to the same expression.]

• Def: (p.441-442 [7]) Let Φ be a smooth k-form.
We say Φ is **closed** if $d_k\Phi = 0$ (i.e. $\Phi \in ker(d_k)$).
We say Φ is **exact** if $\exists\Psi$ such that $\Phi = d_{k-1}\Psi$ (i.e. $\Phi \in im(d_{k-1})$).

Since we know $d_k \circ d_{k-1} = 0$, we have that $im(d_{k-1}) \subseteq ker(d_k)$ for all k, so we get a sequence:

$$\Gamma(\Lambda_0(T\mathcal{M})) \xrightarrow{d_0} \ldots \xrightarrow{d_{k-1}} \Gamma(\Lambda_k(T\mathcal{M})) \xrightarrow{d_k} \Gamma(\Lambda_{k+1}(T\mathcal{M})) \xrightarrow{d_{k+1}} \ldots \xrightarrow{d_{n-1}} \Gamma(\Lambda_n(T\mathcal{M}))$$

of vector spaces and linear maps between them. We compute quotients of subspaces in the nodes of this sequence, called the **de Rham cohomology spaces** (or *de Rham groups* when only referring to the additive structure):

$$H^k_{dR}(\mathcal{M}) := \frac{ker(d_k)}{im(d_{k-1})} = \left\{\Phi + im(d_{k-1}) \;\middle|\; \Phi \in ker(d_k)\right\}.$$

These sequences of groups, $\{H^k_{dR}(\mathcal{M})\}_{k=0}^n$, just described are *homotopy*-invariants (i.e. *homotopically equivalent* manifolds have *isomorphic* de Rham groups) (p.443 [7]). Geometrically speaking, they are supposed to "encode the existence of different dimensional holes in the underlying manifold". This is an avenue to explore in manifold theory. Other than that, knowing whether a given k-form is closed or exact, together with Stoke's Theorem allows you to more easily compute integrals on manifolds with boundary.

<< Chapter IV : Curvature and Geodesics >>

Chapter Menu:

In this chapter, we aim to develop two *systems of differential equations* known as *Geodesic Equations* and *Enstien's Field Equations*. Both use the technology of *connection operators* (i.e. axiomatized directional derivatives on tensor fields) and both are specified uniquely in the presence of a *Riemannian metric* as we will see. We will not be solving these systems in this paper, we leave that to the interested reader as an [Exercise].

Along the way, we pick up the notions of Parallel Transport and Curvature on Manifolds!

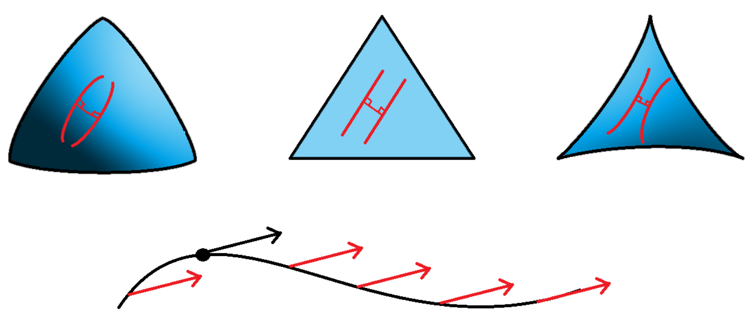


Fig: Intuiting Positive, Nuetral, and Negative Curvature (Top) and Parallel Transport along A Curve (Bottom)

• Def: To avoid confusing new notation in what follows, let us abbreviate our notation for the sets of sections of tensor bundles on a given manifold (i.e. **sets of tensor fields**):

$$\mathcal{T}_l^k(\mathcal{M}) := \Gamma(\mathcal{T}_l^k(T\mathcal{M}))$$

1. Connections on Tensor Bundles

- Def: (p.49-54 [8]) A **connection on the set of all tensor bundles** is a family of maps:

$$\nabla := \left\{ \begin{array}{c} {}^{(k,l)}\nabla : \mathcal{T}^1_0(\mathcal{M}) \times \mathcal{T}^k_l(\mathcal{M}) \to \mathcal{T}^k_l(\mathcal{M}) \\ (X, \Phi) \mapsto {}^{(k,l)}\nabla_X(\Phi) \end{array} \right\}_{(k,l)}$$

collectively satisfying the axioms to follow. Let $N := dim(\mathcal{M})$ and suppose we have a smooth global frame $\{E_1, ..., E_N\}$ for the tangent bundle, then

$\forall f \in C^\infty(\mathcal{M})$,
$\forall \Phi \in \mathcal{T}^k_l(\mathcal{M})$,
$\forall \Psi \in \mathcal{T}^m_n(\mathcal{M})$,

0.) ${}^{(0,0)}\nabla_X f := Xf = X^i E_i f$ [Base Case: Directional Derivative]

1.) ${}^{(1,0)}\nabla_{E_i} E_j := \Gamma^k_{ij} E_k$, [Base Case: Linear Connection]
with the component functions Γ^k_{ij} implicitly satisfying (2-4) in the case of $(k, l) = (1, 0)$.

2.) ${}^{(k,l)}\nabla_{(fX+Y)}\Phi = f \cdot {}^{(k,l)}\nabla_X \Phi + {}^{(k,l)}\nabla_Y \Phi$ [$\mathbb{C}^\infty(\mathcal{M})$-linearity in X]

3.) ${}^{(k,l)}\nabla_X(\lambda\Phi + \Psi) = \lambda \cdot {}^{(k,l)}\nabla_X \Phi + {}^{(k,l)}\nabla_X \Psi$ [$\mathbb{R}$-linearity in Φ]

4.) ${}^{(k+m,l+n)}\nabla_X(\Phi \otimes \Psi) = {}^{(k,l)}\nabla_X(\Phi) \otimes \Psi + \Phi \otimes {}^{(m,n)}\nabla_X(\Psi)$ ["Product Rule"]

$\implies$ 4a.) ${}^{(k,l)}\nabla_X(f\Phi) = Xf \cdot \Phi + f \cdot {}^{(k,l)}\nabla_X \Phi$

5.) $tr({}^{(k,l)}\nabla_X(\Phi)) = {}^{(k,l)}\nabla_X(tr(\Phi))$ [Commutes with Contraction "$tr()$" a.k.a. "$C(\Phi, r, s)$"
(a.k.a. Characteristic Preservation)]

When the valence is understood, we can drop reference to it to simplify the equations. Existence of the general case is implied by existence of linear connections (since we build the definition recursively using the axioms; Existence in the linear case is implied locally in a chart (then globally with partitions of unity) by correct choices of Γ^k_{ij}'s forcing ${}^{(1,0)}\nabla$ to satisfy (2), (3), and (4a).

On the next page, we are going to develop explicit *coordinate expressions* for general connections on each tensor bundle from the axioms. Then we will study *Riemannian connections*, which actually uniquely determine Γ^k_{ij}'s using the metric of $(\mathcal{M}, g)$ and some extra conditions.

1.1 Coordinate Expressions for Each Action of ${}^{(k,l)}\nabla$

We now suppress the indexing for ${}^{(k,l)}\nabla$ and assume that a given chart (U, φ) provides the local coordinate frame $\{E_1, ..., E_n\}|_U := \{\partial_1, ..., \partial_n\}$ as we have seen.

Action of ∇ on $\mathcal{T}^1_0(\mathcal{M})$:

$$\nabla_X Y = \nabla_{X^i \partial_i} Y^j \partial_j = X^i(\nabla_{\partial_i} Y^j \partial_j) \quad \text{[By (2)]}$$

$$= X^i(\partial_i Y^j \partial_j + Y^j \nabla_{\partial_i} \partial_j) \quad \text{[By (4a)]}$$

$$= X^i \partial_i Y^j \partial_j + X^i Y^j \Gamma^k_{ij} \partial_k \quad \text{[Distributing and by (1)]}$$

$$= (XY^k + X^i Y^j \Gamma^k_{ij})\partial_k \quad \text{[Reindexing and Factoring]}$$

$$\therefore \nabla_X Y = (XY^k + X^i Y^j \Gamma^k_{ij})\partial_k \qquad \textbf{(IV.1.1.A)}$$

Action of ∇ on $\mathcal{T}^0_1(\mathcal{M})$:

This one requires a little trickery. Consider (4) applied to $\omega \otimes Y$:

$$\nabla_X(\omega \otimes Y) = \nabla_X(\omega) \otimes Y + \omega \otimes \nabla_X Y$$

Then applying $tr()$ to both sides and using (5) on the left, we get:

$$\nabla_X(tr(\omega \otimes Y)) = tr(\nabla_X(\omega) \otimes Y) + tr(\omega \otimes \nabla_X Y)$$

But for the simple case of contracting a $(1, 1)$ tensor, we have the *natural pairing* notation:

$$\nabla_X\langle\omega, Y\rangle = \langle\nabla_X \omega, Y\rangle + \langle\omega, \nabla_X Y\rangle$$

$$\implies \langle\nabla_X \omega, Y\rangle = \nabla_X\langle\omega, Y\rangle - \langle\omega, \nabla_X Y\rangle$$

And notice that on the right hand side, we can apply (0) and what we just found above in red.

$$\langle\nabla_X \omega, Y\rangle = X(\omega_i Y^i) - \omega_k(XY^k + X^i Y^j \Gamma^k_{ij})$$

$$= X\omega_i Y^i + \omega_i XY^i - \omega_k XY^k - \omega_k X^i Y^j \Gamma^k_{ij} \quad \text{[Product Rule and Distribution]}$$

$$= X\omega_k Y^k - X^i \omega_k \Gamma^k_{ij} Y^j \quad \text{[Reindexing and Commutativity]}$$

$$= (X\omega_k - X^i \omega_j \Gamma^j_{ik})Y^k \quad \text{[Reindexing and Factoring]}$$

$$= \langle(X\omega_k - X^i \omega_j \Gamma^j_{ik})\partial^k, Y\rangle$$

$$\therefore \nabla_X \omega = (X\omega_k - X^i \omega_j \Gamma^j_{ik})\partial^k \qquad \textbf{(IV.1.1.B)}$$

Action of ∇ on $\mathcal{T}^k_l(\mathcal{M})$:

Recall we just found the expressions for the action on the base cases for tensors:

$$\nabla_X Y = (XY^k + X^i Y^j \Gamma^k_{ij})\partial_k$$

$$\nabla_X \omega = (X\omega_k - X^i \omega_j \Gamma^j_{ik})\partial^k.$$

In the special cases where $Y := \partial_{i_a}$ and $\omega := \partial^{j_b}$ these reduce to (re-indexing $p := k$):

$$\nabla_X \partial_{i_a} = X^i \Gamma^p_{i i_a} \partial_p \qquad (\star)$$

$$\nabla_X \partial^{j_b} = -X^i \Gamma^{j_b}_{ip} \partial^p \qquad (\star\star).$$

- -

Let's find the general case for valence (k, l).

$$\nabla_X \Phi = \nabla_X \left(\Phi^{i_1 \dots i_k}_{j_1 \dots j_l} \partial_{i_1} \otimes \dots \otimes \partial_{i_k} \otimes \partial^{j_1} \otimes \dots \otimes \partial^{j_l}\right)$$

Clearly we need to apply the product rule recursively:

$$= \nabla_X (\Phi^{i_1 \dots i_k}_{j_1 \dots j_l}) \partial_{i_1} \otimes \dots \otimes \partial_{i_k} \otimes \partial^{j_1} \otimes \dots \otimes \partial^{j_l}$$

$$+ \Phi^{i_1 \dots i_k}_{j_1 \dots j_l} \nabla_X (\partial_{i_1}) \otimes \dots \otimes \partial_{i_k} \otimes \partial^{j_1} \otimes \dots \otimes \partial^{j_l}$$

$$+ \dots$$

$$+ \Phi^{i_1 \dots i_k}_{j_1 \dots j_l} \partial_{i_1} \otimes \dots \otimes \partial_{i_k} \otimes \partial^{j_1} \otimes \dots \otimes \nabla_X (\partial^{j_l})$$

Now, using (0), $(\star)$, and $(\star\star)$ we have:

$$= X (\Phi^{i_1 \dots i_k}_{j_1 \dots j_l}) \partial_{i_1} \otimes \dots \otimes \partial_{i_k} \otimes \partial^{j_1} \otimes \dots \otimes \partial^{j_l}$$

$$+ \Phi^{i_1 \dots i_k}_{j_1 \dots j_l} (X^i \Gamma^p_{i i_1} \partial_p) \otimes \dots \otimes \partial_{i_k} \otimes \partial^{j_1} \otimes \dots \otimes \partial^{j_l}$$

$$+ \dots$$

$$+ \Phi^{i_1 \dots i_k}_{j_1 \dots j_l} \partial_{i_1} \otimes \dots \otimes \partial_{i_k} \otimes \partial^{j_1} \otimes \dots \otimes (-X^i \Gamma^{j_l}_{ip} \partial^p)$$

- -

Applying multi-linearity and collapsing the definition of Φ:

$$= X\Phi + \left(\Phi^{i_1 \dots i_k}_{j_1 \dots j_l} X^i \Gamma^p_{i i_1}\right) \partial_p \otimes \dots \otimes \partial_{i_k} \otimes \partial^{j_1} \otimes \dots \otimes \partial^{j_l}$$

$$+ \dots - \left(\Phi^{i_1 \dots i_k}_{j_1 \dots j_l} X^i \Gamma^{j_l}_{ip}\right) \partial_{i_1} \otimes \dots \otimes \partial_{i_k} \otimes \partial^{j_1} \otimes \dots \otimes \partial^p$$

(Continues)

Action of ∇ on $\mathcal{T}^k_l(\mathcal{M})$ (Continued):

So far we have:

$$\nabla_X \Phi = X\Phi + \left(\Phi^{i_1 \dots i_k}_{j_1 \dots j_l} X^i \Gamma^p_{ii_1}\right) \partial_p \otimes \dots \otimes \partial_{i_k} \otimes \partial^{j_1} \otimes \dots \otimes \partial^{j_l}$$
$$+ \dots - \left(\Phi^{i_1 \dots i_k}_{j_1 \dots j_l} X^i \Gamma^{j_l}_{ip}\right) \partial_{i_1} \otimes \dots \otimes \partial_{i_k} \otimes \partial^{j_1} \otimes \dots \otimes \partial^p$$

but in order to extract the basis elements, we need to reindex. In each term, we can swap p with the associated i_a or j_b:

$$= X\Phi + \left(\Phi^{p \dots i_k}_{j_1 \dots j_l} X^i \Gamma^{i_1}_{ip}\right) \partial_{i_1} \otimes \dots \otimes \partial_{i_k} \otimes \partial^{j_1} \otimes \dots \otimes \partial^{j_l}$$
$$+ \dots - \left(\Phi^{i_1 \dots i_k}_{j_1 \dots p} X^i \Gamma^p_{ij_l}\right) \partial_{i_1} \otimes \dots \otimes \partial_{i_k} \otimes \partial^{j_1} \otimes \dots \otimes \partial^{j_l}$$

Finally, factoring out the basis elements we get (afterwards we'll call $i_p := p$ or $j_p := p$)::

$$\nabla_X \Phi = \left(X\Phi^{i_1 \dots i_k}_{j_1 \dots j_l} + \sum_{a=1}^{k} \Phi^{i_1 \dots p \dots i_k}_{j_1 \dots j_l} X^i \Gamma^{i_a}_{ip} - \sum_{b=1}^{l} \Phi^{i_1 \dots i_k}_{j_1 \dots p \dots j_l} X^i \Gamma^p_{ij_b} \right) \partial_{i_1} \otimes \dots \otimes \partial^{j_l} \tag{IV.1.1.C}$$

Notes:

i.) These calculations hold equally well in a global frame $\{E_1, \dots, E_n\}$ for the tangent bundle. We restricted to a chart here for existence proof reasons. Take these and sum over a partition of unity for the global expression with respect to an atlas. For said frame $\{E_i\}$ equation, just replace the appropriate basis elements.

ii.) Again, the Γ^k_{ij}'s are variables that are assumed to make the linear connection ${}^{(1,0)}\nabla$ satisfy (2), (3), and (4a). So there is a set of connections possible on the set of all tensor bundles for an arbitrary smooth manifold:

$$\mathfrak{Conn}(\mathcal{M}) := \left\{ \nabla \;\middle|\; \nabla \text{ is determined by } \Gamma^k_{ij}\text{'s which satisfy (2),(3), and (4a)} \right\}$$

iii.) [Exercise: To complete the existence proof, show the **Euclidean Connection**, given by $\Gamma^k_{ij} = 0$ for all indices satisies the required axioms.]

1.2 Riemannian Connections - Determining Γ^k_{ij}'s with $g(X, Y)$

• Def: (p.65-70 [8]) Given a *Riemannian Manifold* $(\mathcal{M}, g)$, we define a **Riemannian Connection**, also denoted ∇, to be a *connection on the set of all tensor bundles* subject to the following:

6.) $\nabla_X\big(g(Y, Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ [Compatibility with g]

7.) $\nabla_X Y - \nabla_Y X = XY - YX$ [Symmetry]

• Prop: **(Riemannian Connection Coefficients):**

The linear coefficients, Γ^k_{ij}'s, for a Riemannian connection ∇ are uniquely determined by the metric via the local formulas:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\big(\partial_i g_{jl} - \partial_l g_{ij} + \partial_j g_{li}\big) \qquad \textbf{(IV.1.2)}$$

where g^{kl} represents the coefficients of the inverse matrix for the metric–recall II.3.2.

• Lemma: $\partial_i \partial_j - \partial_j \partial_i = 0$ (equality as linear operators on $C^\infty(U)$). [Exercise: Prove this.]

Proof: Consider (7) in the special case of $X = \partial_i$, $Y = \partial_j$. We get:

$$\nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i = \partial_i\partial_j - \partial_j\partial_i.$$

$\implies \Gamma^k_{ij}\partial_k - \Gamma^k_{ji}\partial_k = (\Gamma^k_{ij} - \Gamma^k_{ji})\partial_k = 0$ [By definition of $\nabla_{\partial_i}\partial_j$ and the Lemma]

$\implies \Gamma^k_{ij} = \Gamma^k_{ji}$ for all triples of indices (i, j, k).

- -

Next, let's derive the coordinate expression of (6) with $X = \partial_i$, $Y = \partial_j$, and $Z = \partial_k$.

$\nabla_{\partial_i}\big(g(\partial_j, \partial_k)\big) = g(\nabla_{\partial_i}\partial_j, \partial_k) + g(\partial_j, \nabla_{\partial_i}\partial_k)$

$\implies \partial_i g_{jk} = g(\Gamma^l_{ij}\partial_l, \partial_k) + g(\partial_j, \Gamma^l_{ik}\partial_l)$

$= g_{lk}\Gamma^l_{ij} + g_{jl}\Gamma^l_{ik}$

Easy enough, now let's rewrite this last result 3 times with the indices cyclically permuted:

$$I.) \qquad \partial_i g_{jk} = g_{lk}\Gamma^l_{ij} + g_{jl}\Gamma^l_{ik}$$

$$II.) \qquad \partial_k g_{ij} = g_{lj}\Gamma^l_{ki} + g_{il}\Gamma^l_{kj}$$

$$III.) \qquad \partial_j g_{ki} = g_{li}\Gamma^l_{jk} + g_{kl}\Gamma^l_{ji}$$

Since we have coefficient symmetry in both the metric and the gammas, the matching colors on the right hand side are equal. Hence (I-II+III) gives:

$$\boldsymbol{\partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ki} = 2 g_{lk} \Gamma^l_{ij}}$$

from which the result is clear swapping $\boldsymbol{k}$ and $\boldsymbol{l}$ indices and multiplying both sides by $\boldsymbol{\frac{1}{2} g^{lk}}$ (recall also $\boldsymbol{g}$ is symmetric). ■

To conclude this section, note the following:

★ Prop: Every smooth manifold admits a smooth Riemannian metric (induced from the charts and put together by partition of unity – the result happens to be symmetric and positive-definite). Hence, by the above work, on any given smooth manifold we have a uniquely determined Riemannian Connection. [See Thm 4.5, p.193 and Thm 3.3, p.314 [11]]. ■

Because of this result, in the sequel we will assume the context of a Riemannian metric, $\boldsymbol{g(\cdot, \cdot\cdot)}$ *and a Riemannian connection* $\boldsymbol{\nabla}$.

2. Covariant Derivatives, Geodesic Eq's, & Parallel Transport

2.1 Covariant Derivatives

• Def: (p.50&54 [8]) Recall for a *connection on the set of all tensor bundles*, $\boldsymbol{\nabla}$, we have the maps:

$$^{(k,l)}\boldsymbol{\nabla} : \mathcal{T}^1_0(\mathcal{M}) \times \mathcal{T}^k_l(\mathcal{M}) \to \mathcal{T}^k_l(\mathcal{M}).$$

For given fields $\boldsymbol{X}$ and $\boldsymbol{\Phi}$, we call $\boldsymbol{\nabla_X \Phi} \in \mathcal{T}^k_l(\mathcal{M})$ the **covariant derivative of $\boldsymbol{\Phi}$ w.r.t. $\boldsymbol{X}$**.

• Def: If we instead consider a given tensor field $\boldsymbol{\Phi}$ with $\boldsymbol{X}$ deferred to be another argument, we obtain the maps (overloading the symbols):

$$^{(k,l)}\boldsymbol{\nabla} : \mathcal{T}^k_l(\mathcal{M}) \to \mathcal{T}^k_{l+1}(\mathcal{M})$$

$$\left[{}^{(k,l)}\boldsymbol{\nabla}(\boldsymbol{\Phi})\right](\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_k, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_l, \boldsymbol{X}_{l+1}) := \left[{}^{(k,l)}\boldsymbol{\nabla}_{\boldsymbol{X}_{l+1}}\boldsymbol{\Phi}\right](\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_k, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_l)$$

and we call $\boldsymbol{\nabla\Phi} \in \mathcal{T}^k_{l+1}$ the **total covariant derivative of $\boldsymbol{\Phi}$**. In this case, we have another notation that we use to denote the coordinate functions of $\boldsymbol{\nabla\Phi}$:

$$(\boldsymbol{\nabla\Phi})^{i_1 \ldots i_k}_{j_1, \ldots, j_l;\, j_{l+1}} := (\boldsymbol{\nabla\Phi})^{i_1 \ldots i_k}_{j_1, \ldots, j_l j_{l+1}}$$

that is, we indicate the w.r.t. variable separated by a semicolon. In a local frame, this reads:

$$\boldsymbol{\nabla\Phi} = \Big((\boldsymbol{\nabla\Phi})^{i_1 \ldots i_k}_{j_1, \ldots, j_l;\, j_{l+1}}\Big)\boldsymbol{\partial}_{i_1} \otimes \ldots \otimes \boldsymbol{\partial}^{j_l} \otimes \boldsymbol{\partial}^{j_{l+1}}$$

(Continues)

2.2 Geodesic Equations

• Def: (p.55 [8]) We define a **curve in $\mathcal{M}$** to be a (smooth) map, $\gamma : I \to \mathcal{M}$, with the particular case of a **curve segment with endpoints** having $I := [0, 1] \subset \mathbb{R}$. Regularity varies among contexts.

Notes:
1.) Some would argue that $\gamma : I \to \mathcal{M}$ defines a *path*, whereas the *curve* is the geometric object specified by $[\gamma]_\sim$, the equivalence class of paths under *re-parameterization.*

2.) Given a particular parameterization, $\gamma(t)$, we have a notion of *evolution* or *trajectory* along the curve by varying t - hence a notion of *velocity* and *acceleration* as well.

• Def: (p.56 [8]) For an *injective* curve $\gamma : I \to \mathcal{M}$, we define its **velocity field**, to be the push-forward of the "time" derivative. That is, the vector field $\dot\gamma \in \mathcal{T}_0^1(Im(\gamma))$ is given by its action on $f \in C^\infty(Im(\gamma))$:

$$\dot\gamma_{\gamma(t_0)}(f) := \left[d\gamma_{t_0}(\partial_t\big|_{t_0})\right](f)$$

at each point $t = t_0$. We use the notation $\dot\gamma(t) := \dot\gamma_{\gamma(t)}$ as well.

- -

By the discussion leading up to equation II.2.A, in an image chart (V, ψ), we have a coordinate/basis representation:

$$[\dot{\widehat{\gamma}}(t)]_{\beta_2} = \begin{bmatrix} \partial_t \widehat{\gamma^1}(t) \\ \vdots \\ \partial_t \widehat{\gamma^n}(t) \end{bmatrix}$$

where of course $\widehat{\gamma} := \psi \circ \gamma : I \to \mathbb{R}^n$. With some notation suppressed:

$$\dot\gamma(t) = (\dot\gamma^1(t), ..., \dot\gamma^n(t))^t = (\partial_t \gamma^k)\partial_k.$$

• Def: (p.57-58 [8]) In the presence of a *connection*, ∇, we define γ's **acceleration field** as:

$$\nabla_{\dot\gamma(t)}(\dot\gamma(t)) \in \mathcal{T}_0^1(Im(\gamma)),$$

which by (IV.1.1.A) has local coordinate expression (t suppressed):

$$\nabla_{\dot\gamma}(\dot\gamma) = (\dot\gamma\dot\gamma^k + \dot\gamma^i\dot\gamma^j\Gamma_{ij}^k)\partial_k$$

• Def: (p.58 [7]) A **geodesic** is a curve $\gamma : I \to \mathcal{M}$ whose acceleration field is identically zero in every chart (i.e. $\nabla_{\dot\gamma}(\dot\gamma) = 0$). The above equation says a curve is a geodesic iff:

$$\forall (V, \psi), \forall k, \qquad \dot\gamma\dot\gamma^k + \dot\gamma^i\dot\gamma^j\Gamma_{ij}^k = 0 \qquad \textbf{(IV.2.2)}$$

This system of second order ODE's (in the variable t) is referred to as the **geodesic equations**.

2.3 Parallel Transport

We saw previously how an *injective* and *at least* $C^2(I)$ curve $\gamma : I \to \mathcal{M}$ gives rise to its *velocity* $\dot{\gamma} \in \mathcal{T}_0^1(Im(\gamma))$ and *acceleration fields* $\nabla_{\dot{\gamma}}(\dot{\gamma}) \in \mathcal{T}_0^1(Im(\gamma))$, we saw their local coordinate expressions, and we utilized the notion of *zero acceleration* to get the *geodesic equations*.

Now, we wish to use this same notion of *zero acceleration* to express the *covariant derivative of a tensor field* Φ *along this* γ in a different form.

The following generalizes the discussion of (p.59-62 [8]) and instantiates the discussion in [15] to the case of connections on tensor bundles.

• Def/Prop: Let Φ_p be a tensor at the point $p \in Im(\gamma)$ for an injective, smooth curve $\gamma : I \to \mathcal{M}$ contained in a single chart (V, ψ) [Exercise: Generalize to multiple charts later]. We can define a unique new tensor field, $\Psi \in \mathcal{T}_l^k(Im(\gamma))$, along the curve such that:

$$\nabla_{\dot{\gamma}}(\Psi) = 0 \quad \text{and} \quad \Psi_p := \Phi_p. \tag{IV.2.3}$$

The solution to the above *initial value* system of ODE's defines Ψ_q at the other points $q \in Im(\gamma)$.

Using this Ψ, and setting $p = \gamma(t_0)$, we can define a family of maps indexed by *start* and *stop* points (t_0, t_1):

$$P^{\gamma}_{t_0 t_1} : \mathcal{T}_l^k(T_{\gamma(t_0)} Im(\gamma)) \to \mathcal{T}_l^k(T_{\gamma(t_1)} Im(\gamma))$$
$$P^{\gamma}_{t_0 t_1}\left(\Phi_{\gamma(t_0)}\right) := \Psi_{\gamma(t_1)}$$

which are *linear isomorphisms* that are dependent on γ (see the ODE's).
We refer to this family as the **parallel translations of $\Phi_{\gamma(t_0)}$ along γ**.

Proof: [Exercise: Find the explicit solution Ψ (hence the explicit form of the $P^{\gamma}_{t_0 t_1}$'s).
Use Picard's Existence Theorem [15].]

• Prop: With the parallel translation family defined above, we have the following correspondence:

$$\left(\nabla_{\dot{\gamma}}\Phi\right)_{t_0} = \lim_{h \to 0} \frac{1}{h}\left(P^{-1}_{t_0(t_0+h)}(\Phi_{\gamma(t_0+h)}) - \Phi_{\gamma(t_0)}\right) = \partial_t\big|_{t=0}\left(P^{\gamma}_{t_0(t_0+t)}(\Phi_{\gamma(t)})\right)$$

Proof: [Exercise: Prove this! Use eq. (IV.1.1.C) and your result from the previous proof.]

---

3. Curvature, Tensor Fields, and Einstein's Equations

The following is based off of [8, 11, 13], see Introduction. Care must be taken when applying to the psuedo-Riemannian case! (See p.131-132 [8]).

3.1: Curvature

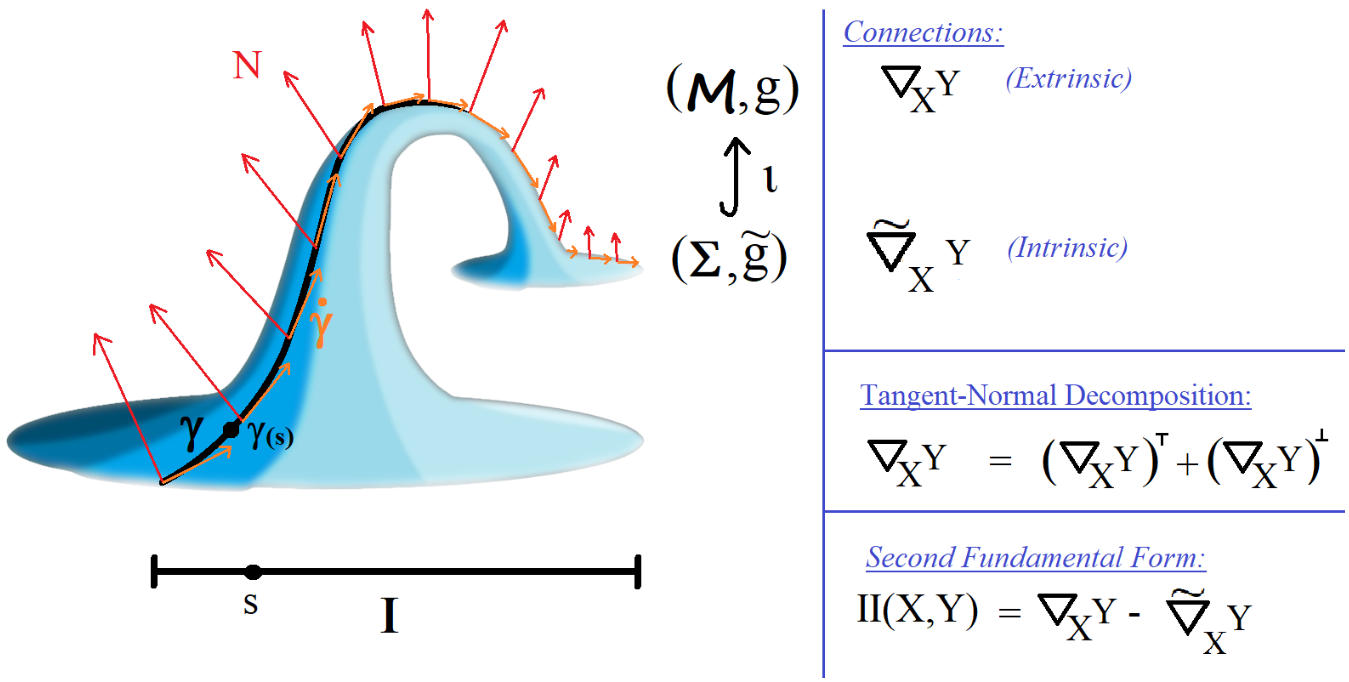


Fig: Visualizing Extrinsic vs. Intrinsic Curvature of Curves Embedded in Riemannian Submanifolds

---

Let's begin with defining a *Riemannian submanifold.*

• Def: (p.333 [7]) Let $(\Sigma, \tilde{g})$ and $(\mathcal{M}, g)$ be Riemannian manifolds. If there exists a smooth *immersion* or *embedding* $\iota : \Sigma \to \mathcal{M}$, that is also an isometry (i.e. $\iota^* g = \tilde{g}$), then we say Σ is a **Riemannian submanifold** of $\mathcal{M}$.

• Def: By way of the differential of the "inclusion" map $d\iota : T\Sigma \to T\mathcal{M}$, we get an *orthogonal complement decomposition* of the tangent bundle over $Im(\iota) = \iota(\Sigma)$:

$$T(\iota(\Sigma)) = d\iota(T\Sigma) \oplus \big(d\iota(T\Sigma)\big)^{\perp}. \qquad (\star)$$

where, we define the **orthogonal complement projection** as usual:

$$V^{\perp} := \big\{ w \in W \mid \langle w, v \rangle_g = 0,\ \forall v \in V \big\}.$$

The decomposition given above can be loosely referred to as the **tangent-normal decomposition**.

Considering $(\bigstar)$ again, we get an associated splitting of the sections into tangent and normal parts as well, giving rise to the definition of **normal vector fields**, $\boldsymbol{N}$, as sections of the orthogonal complement of the tangent bundle. Depending on the "inclusion" map (i.e. depending on the rank of its differential), we define the **codimension of $\Sigma \subseteq \mathcal{M}$** to be the quantity:

$$\boldsymbol{codim}(\Sigma) := \boldsymbol{dim}\big(T_p(\iota(\Sigma))\big) - \boldsymbol{dim}\big(d\iota(T_p\Sigma)\big) = \boldsymbol{dim}\big((d\iota(T_p\Sigma))^{\perp}\big)$$

at any point $p \in \mathcal{M}$ since vector bundles have consistent fibral dimension at each point.

Now, consider a given Riemannian submanifold $(\Sigma, \widetilde{g}) \subseteq (\mathcal{M}, g)$. As independent Riemannian manifolds, they both have their own linear connections (respectively: $\widetilde{\nabla}$ and ∇) defined by their metrics $\widetilde{g}$ and g [Recall formula (IV.1.2)].

Decomposing the image of the *ambient connection* $\nabla_X Y$ over $\iota(\Sigma)$, we get:

$$\nabla_X Y = (\nabla_X Y)^{\top} + (\nabla_X Y)^{\perp}. \qquad (\bigstar\bigstar)$$

It can be shown by a uniqueness argument that over $\iota(\Sigma)$, $(\nabla_X Y)^{\top} = \widetilde{\nabla}_X Y$.

[Exercise: Prove this! See (p.135 [8])].

• Def: (p.134 [8]) We rename the normal projection of the image of the ambient connection restricted to $\iota(\Sigma)$:

$$II(X,Y) := (\nabla_X Y)^{\perp}$$

and call it the **Second Fundamental Form**. With $(\bigstar\bigstar)$ and the last exercise, we may write:

$$II(X,Y) := \nabla_X Y - \widetilde{\nabla}_X Y$$

meaning the second fundamental form measures the *difference between the images of the ambient and induced connections.*

• Def: (p.137 [8]) Given a Riemanian submanifold $(\Sigma, \widetilde{g}) \subseteq (\mathcal{M}, g)$, we define the **(extrinsic) curvature of an embedded curve $\gamma : I \to \Sigma$** as the *magnitude of the g-acceleration* and the **(intrinsic) curvature of $\gamma : I \to \Sigma$** as the *magnitude of the $\widetilde{g}$-acceleration* (in both cases with γ parameterized by arc-length s).

Respectively listed in symbols, this is just:

$$\kappa_{ext}(s) := \Big|\nabla_{\dot\gamma(s)}\dot\gamma(s)\Big|_g \qquad \text{and} \qquad \kappa_{int}(s) := \Big|\widetilde{\nabla}_{\dot\gamma(s)}\dot\gamma(s)\Big|_{\widetilde{g}}$$

Thus, the second fundamental form can be used to analyze the difference in curvatures for γ.

3.2 Riemannian Curvature Tensor Fields

In the last section, we did a lot of work characterizing intrinsic and extrinsic curvature of embedded curves in Riemannian submanifolds. We don't always have an ambient manifold to work with and so are stuck with the intrinsic definition in the general case. So we aren't concerned apriori with the second fundamental form.

However, we want to define a curvature operator using $\boldsymbol{\nabla}$, that is not dependent on any one particular curve $\boldsymbol{\gamma}$, but rather encodes the information for all curves into the tangent spaces $\boldsymbol{Tp\mathcal{M}}$ at each point (or better yet, into the sections of the tangent bundles $\mathcal{T}_0^1(\boldsymbol{\mathcal{M}})$).

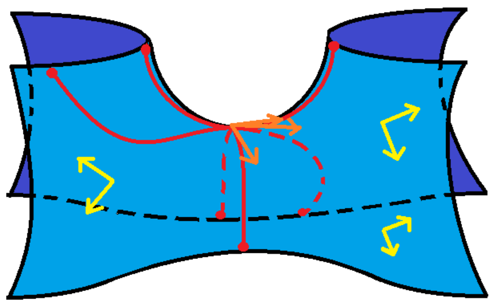


Fig: Brainstorming

Now, we have $\boldsymbol{\nabla_X Y}$ at our disposal and we want to characterize curvature axiomatically in terms of these symbols. The following is based on (p.115-117 [8]).

• Def: **(The Flatness Criterion)**: We say a manifold is **flat** if its connection operator satisfies the following forall $\boldsymbol{X, Y, Z} \in \mathcal{T}_0^1(\boldsymbol{\mathcal{M}})$:

$$\boldsymbol{\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) = \nabla_{[X,Y]}(Z)}$$

where $\boldsymbol{[X, Y] := XY - YX}$.

Notes: This definition was motivated from the **Euclidean connection**, $\boldsymbol{\overline{\nabla}_X Y := XY}$, which obviously satisfies the flatness criterion. Writing out the coordinate expressions for each side, one will see that this equality is wildly not held in general. However, any manifold *isometric* to Euclidean space will satisfy this equation [Exercise: Prove this!] and so we use it to define:

• Def: (p.117 [8]) The **Riemannian Curvature Endomorphism**, which measures the deviation of the *flatness criterion* to be held, is given by:

$$\boldsymbol{R(\#1, \#2)\#3 : \mathcal{T}_0^1(\mathcal{M}) \times \mathcal{T}_0^1(\mathcal{M}) \times \mathcal{T}_0^1(\mathcal{M}) \to \mathcal{T}_0^1(\mathcal{M})}$$

$$\boldsymbol{R(X, Y)Z := \left[\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right](Z)}$$

The remainder of this subsection is dedicated to creating a tensor field out of the *Riemannian Curvature Endomorphism* defined above, providing its component functions in a local frame, and listing a few of its properties.

★ Def: (p.118 [8]) The **Riemannian Curvature Tensor Field** is created from the endomorphism by composing with the metric (as we have done before in Section II.3 raising and lowering indices):

$$R(X,Y,Z,W) := g(R(X,Y)Z,W)$$

Rewriting the definition, we've obtained a map:

$$R : \mathcal{T}^1_0(\mathcal{M})^4 \to C^\infty(\mathcal{M})$$
$$R(X,Y,Z,W) := g(\left[\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\right](Z), W)$$

which is multilinear over $C^\infty(\mathcal{M})$ in each field variable [Exercise: Prove this!], so it defines a valence $(0,4)$ tensor field.

★ Prop: **(Coordinate Expression for $R(X,Y,Z,W)$):**

In the standard frame for a chart, the Riemannian Curvature Tensor is given by:

$$R = R_{ijkl}\partial^i \otimes \partial^j \otimes \partial^k \otimes \partial^l$$

where $R_{ijkl} := R(\partial_i, \partial_j, \partial_k, \partial_l) = g_{ml}\left(\partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik} + \Gamma^a_{jk}\Gamma^m_{ia} - \Gamma^a_{ik}\Gamma^m_{ja}\right)$.

Proof: [Exercise: Easy! Go compute. Use $\nabla_{\partial_a}\partial_b = \Gamma^c_{ab}\partial_c$ and connection axioms.] ■

• Prop: **(Properties of $R(X,Y,Z,W)$):**

(p.122 [8]) The Riemann curvature tensor field satisfies the following component equations:
1.) $R_{ijkl} = -R_{jikl}$
2.) $R_{ijkl} = -R_{ijlk}$
3.) $R_{ijkl} = R_{klij}$
4.) $R_{ijkl} + R_{jkil} + R_{kijl} = 0$

Proof: [Exercise: Use the above proposition.] ■

• Def: **(Contracted Forms of $R(X,Y,Z,W)$):**
(p.124 [8]) The **Ricci Curvature Tensor Field** is given by contracting the Riemannian Curvature Tensor on its first and last indices. The **Scalar Curvature** is the subsequent contraction of the Ricci Curvature on its only two indices.

3.3: Einstein's Field Equations

• Def: Although Lee does discuss this topic (see p.125-126 [8]), we switch texts now to Wald's text on *General Relativity* (p.72-73 [14]), wherein it is stated that:

> *"The entire content of general relativity may be summarized as follows: Spacetime is a manifold $\mathcal{M}$ on which there is defined a Lorentz metric g_{ab}. The curvature of g_{ab} is related to the matter distribution in spacetime by* ***Einstein's equation****..."*

$$R_{ab} - \frac{1}{2}R \cdot g_{ab} = 8\pi \cdot T_{ab}(g_{cd}) \qquad \textbf{(IV.3.3.A)}$$

where R_{ab} and R in his notation stand for the *Ricci* and *Scalar Curvatures* defined above and T_{ab} is the **Stress tensor**.

Notes: On (p.73 [14]), the author makes two main points concerning these equations, which are of interest to us immediately:

1.) "Einstein's equation is equivalent to a *coupled system of nonlinear, second-order partial differential equations* for the *metric components* $g_{\mu\nu}$. For a metric of *Lorentz signature*, these equations have a hyperbolic (i.e., wave equation) character..." [Exercise: Prove this!].

2.) "Until g_{ab} is known, we do not know how to physically interpret T_{ab}... thus in general relativity, one must solve simultaneously for the spacetime metric and the matter distribution."

For solution methods to this system, see Ch.7 [14]... (really the entirety of his text is worth reading as a continuation). I leave you with a fully written out version of the above equations in a chart, with the Christoffel symbols replaced by (IV.1.2) - which doesn't depend on positive definite vs. nondegenerate so psuedo-Riemannian metrics like the Lorentz metric still give the same symbols by virtue of the symmetry and compatibility conditions.

(Continues)

Einstein's Equations (In Coordinates):

Let's summarize what we have in a chart (U, φ), with local frame $\partial_i := (\varphi^{-1})_*(\frac{\partial}{\partial x^i})$:

1.) $R_{jk} - \frac{1}{2}Rg_{jk} = 8\pi T_{jk}$ [Master Equation (IV.3.3.A)]

2.) $R_{ijkl} := g_{ml}\left(\partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik} + \Gamma^a_{jk}\Gamma^m_{ia} - \Gamma^a_{ik}\Gamma^m_{ja}\right)$ [Riemannian Curvature Tensor Coeff.]

3.) $R_{jk} = \partial_i\Gamma^i_{jk} - \partial_j\Gamma^i_{ik} + \Gamma^a_{jk}\Gamma^i_{ia} - \Gamma^a_{ik}\Gamma^i_{ja}$ [Ricci Tensor: $R_{jk} := g^{li}R_{ijkl}$]

4.) $R = g^{kj}\left(\partial_i\Gamma^i_{jk} - \partial_j\Gamma^i_{ik} + \Gamma^a_{jk}\Gamma^i_{ia} - \Gamma^a_{ik}\Gamma^i_{ja}\right)$ [Scalar Curvature: $R := g^{kj}R_{jk}$]

5.) $\Gamma^c_{ab} = \frac{1}{2}g^{cd}\left(\partial_a g_{bd} - \partial_d g_{ab} + \partial_b g_{da}\right)$ [(IV.1.2)]

Now, if we carefully consider the indices of each of the Γ^c_{ab}'s appearing in (3) and (4), we may substitute (5) into (3) and (4), subsequently substituting those into the master equation (1). After accomplishing this, we will have the explicit system in terms of only the metric and stress coefficients. The results are provided for this calculation below:

$$\Bigg\{\Big[\partial_i[\tfrac{1}{2}g^{il}(\partial_j g_{kl} - \partial_l g_{jk} + \partial_k g_{lj})] - \partial_j[\tfrac{1}{2}g^{il}(\partial_i g_{kl} - \partial_l g_{ik} + \partial_k g_{li})] + [\tfrac{1}{2}g^{ml}(\partial_j g_{kl} - \partial_l g_{jk} + \partial_k g_{lj})][\tfrac{1}{2}g^{il}(\partial_i g_{ml} - \partial_l g_{im} + \partial_m g_{li})] - [\tfrac{1}{2}g^{ml}(\partial_i g_{kl} - \partial_l g_{ik} + \partial_k g_{li})][\tfrac{1}{2}g^{il}(\partial_j g_{ml} - \partial_l g_{jm} + \partial_m g_{lj})]\Big]$$

$$-\tfrac{1}{2}g^{kj}\Big[\partial_i[\tfrac{1}{2}g^{il}(\partial_j g_{kl} - \partial_l g_{jk} + \partial_k g_{lj})] - \partial_j[\tfrac{1}{2}g^{il}(\partial_i g_{kl} - \partial_l g_{ik} + \partial_k g_{li})] + [\tfrac{1}{2}g^{ml}(\partial_j g_{kl} - \partial_l g_{jk} + \partial_k g_{lj})][\tfrac{1}{2}g^{il}(\partial_i g_{ml} - \partial_l g_{im} + \partial_m g_{li})] - [\tfrac{1}{2}g^{ml}(\partial_i g_{kl} - \partial_l g_{ik} + \partial_k g_{li})][\tfrac{1}{2}g^{il}(\partial_j g_{ml} - \partial_l g_{jm} + \partial_m g_{lj})]\Big]g_{jk}$$

$$= 8\pi T_{jk}(g_{11}, \ldots, g_{nn})\Bigg\}_{j,k\in\{1,\ldots,n\}}$$

(IV.3.3.B)

[6:22 pm GMT-8, 01.05.2021. KTS]

Appendix A. Notation Summary

Most of the time, we will be writing basis expansions of tensor fields in terms of a single chart $(U, \varphi : U \subseteq \mathcal{M} \to \mathbb{R}^n)$, with coordinate vector and covector field bases $\{\partial_1, \ldots, \partial_n\}$ and $\{\partial^1, \ldots, \partial^n\}$, where $\partial_i|_p := \varphi_*(\partial_i|_{\varphi(p)})$ and likewise $\partial^i|_p := \varphi^*(\partial^i|_{\varphi(p)})$. Thus for a (k, l) tensor field, we get the expansion:

$$\Phi = \Phi^{i_1,\ldots,i_k}_{j_1\ldots,j_l} \cdot \partial_{i_1} \otimes \ldots \otimes \partial_{i_k} \otimes \partial^{j_1} \otimes \ldots \otimes \partial^{j_l}.$$

which is implicitly a point-dependent expression over U. If one is given a collection of global fields constituting a frame, then one may have an expansion that does not depend on the chart:

$$\Phi = \Phi^{i_1,\ldots,i_k}_{j_1\ldots,j_l} \cdot E_{i_1} \otimes \ldots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \ldots \otimes \varepsilon^{j_l}$$

for an example. We do not really use the latter notation however. In most cases when global constructs are needed, we patch together local ones with partitions of unity $\{f_\alpha : U_\alpha \to \mathbb{R}\}_{\alpha \in A}$ to be described.

The base cases for tensor fields are valence $(1, 0)$ and $(0, 1)$ (respectively vector and co-vector fields), denoted locally by:

$$X = X^i \partial_i \qquad \text{and} \qquad \omega = \omega_i \partial^i.$$

Some other notations used for the basis fields are $\frac{\partial}{\partial x^i} := \partial_i =: e_i$ and $dx^i := \partial^i =: e^i$.

There is ambiguity sometimes in writing out the coordinate expression for a tensor field, since one writes the same symbol for the partials. Pay attention to the point dependency if it is specified, this is more or less important when talking about the component functions, which are hence either $\Phi^{i_1\ldots i_k}_{j_1\ldots j_l} : U \to \mathbb{R}$ or $\Phi^{i_1\ldots i_k}_{j_1\ldots j_l} \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$ depending on context. One might see just a real coordinate expression with the chart implied via pullback notation or words.

To make clear throughout, I specify valence and symmetry type in set theoretic notation e.g.:

$$\Gamma(\Xi^k_l(T\mathcal{M})) \subseteq \Gamma(\mathcal{T}^k_l(T\mathcal{M})) \supseteq \Gamma(\Lambda^k_l(T\mathcal{M}))$$

$$\text{Symmetric} \subseteq \text{Regular} \supseteq \text{Skew-symmetric}$$

In particular, for differential forms $\Phi \in \Gamma(\Lambda_k(T\mathcal{M})) =: \Omega^k(\mathcal{M})$ we have strictly ordered multi-index notation $I = (i_1 \ldots i_k)$ for the basis vectors of the alternating tensor spaces at each point given by the elementary k-forms:

$$\partial^I = \partial^{i_1} \wedge \ldots \wedge \partial^{i_k}.$$

This is also seen as $dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ etc.

Certain quantities do not obey the indexing conventions as tensors do for example the exterior derivative operators I've lower indexed. They have to be indexed somehow! It seems most authors just write d to ambiguously mean the family of d_k's.

In other texts, one has the so called "mu-nu" notation, which instead of set membership notation uses latin indices to indicate valence and greek indices to indicate basis expansions in those same indices listed. For example:

$$T^a_b = T^\mu_\nu \partial_\mu \otimes \partial^\nu$$

for a valence $(\mathbf{1}, \mathbf{1})$ tensor field. There is a combination of notations apparently, let index behavior guide you. Contra-variant component indices up, Co-variant component indices down.

Texts like [14] in particular also use a shorthand for the *symmetrization* and *skew-symmetrization* of tensors across selected indices, respectively written:

$$\boldsymbol{T_{(ab)} := Sym(T_{ab})} \qquad \text{and} \qquad \boldsymbol{T_{[ab]} := Alt(T_{ab})}$$

for example.

Appendix B: Abstractions of Tensors

The discussion below is based off of multiple sources. See here for overall citations.
Import Abstract Algebra.

Multilinear Maps on a Single (Real) Vector Space

In the beginning of this document, before the development of tensor fields over manifolds with the idea of sections of vector/fiber bundles, there were just plain old tensors on a (single) vector space.

$$\mathcal{T}_l^k(V) := \left\{\varphi : (V^*)^k \times V^l \to \mathbb{R} \;\middle|\; \varphi \text{ is linear in each variable}\right\}$$

and we had the basis expansions (assuming a canonical basis existed):

$$\varphi = \varphi^{i_1 \dots i_k}_{j_1 \dots j_l} \partial_{i_1} \otimes \dots \otimes \partial_{i_k} \otimes \partial^{j_1} \otimes \dots \otimes \partial^{j_l}$$

written in terms of this curious new symbol $\otimes$ that essentially welded together existing linear functions in the base cases and created multilinear ones. That is,

$$(\varphi, \psi) \mapsto \varphi \otimes \psi.$$

Multilinear Maps on Different (Arbitrary) Vector Spaces or Modules

Although we stuck to a single vector space and its dual, there was nothing stopping us from welding together linear functions defined on any number of different (real) vector spaces of potentially different (finite) dimensions:

$$\varphi : V_1^{n_1} \times \dots \times V_m^{n_m} \to V_{m+1}^{n_{m+1}}$$

where V_i may have no relation to V_j other than the scalar field. Moreover, in general for abstract vector spaces, there is no canonical basis $\{\partial_1, \dots, \partial_{n_i}\}$. Why we got away with this before is that our charts go into $\mathbb{R}^n$ which we know has a canonical basis. And we get the induced basis on the tangent spaces etc. using the pushforward/pullback mechanisms.

We also can generalize to complex vector spaces, vector spaces over arbitrary fields (V/F), or even to modules over rings (${}_RV$ or V_R) instead, but then we get sidedness happening and

$$\varphi \otimes \psi(v, w) = \varphi(v) \cdot \psi(w)$$

may start failing to be multi-linear if we can't transfer the scalars all to one side (i.e. if $\psi(\lambda w) = \lambda\psi(w)$ but $\varphi(v)\lambda \neq \lambda\varphi(v)$). There is a relaxed notion called *middle-linearity*, but this is usually discussed in a binary context not in general.

(Continues)

Abstract Tensor Products of Modules

• Def: Let $V_1, ..., V_m$ be R-modules over a *commutative ring with unit* R. The **free R-module over the cartesian product** ($V := V_1 \times ... \times V_m$) is defined as:

$$\mathcal{F}(V_1 \times ... \times V_m) := \left\{ \text{finite formal } R\text{-linear combinations of } (v_1, ..., v_m) \;\middle|\; v_i \in V_i \right\}$$

together with addition and the left R-action specified by:

$$\sum_{v \in V} \lambda_v v + \sum_{v \in V} \mu_v v := \sum_{v \in V} (\lambda_v + \mu_v) v$$

and

$$\mu * \left(\sum_{v \in V} \lambda_v v \right) := \sum_{v \in V} (\mu \cdot \lambda_v) v$$

[Exercise: Check that the module axioms are satisfied by these (see p.337 [3]).]

• Def: Now let $\mathcal{S}$ be the *submodule* generated by elements of the form:

$$v = (v_1, ..., v_i + v_i', ..., v_m) - (v_1, ..., v_i, ..., v_m) - (v_1, .., v_i', ..., v_m)$$

or

$$v = (v_1, ..., \lambda v_i, ..., v_m) - \lambda(v_1, ..., v_i, ..., v_m)$$

for arbitrary scalar $\lambda \in R$, component index $i \in \{1, ..., m\}$, and elements $v_j \in V_j$.

• Def: We define the **tensor product of the R-modules** $V_1, ..., V_m$ by the quotient module:

$$V_1 \otimes ... \otimes V_m := \mathcal{F}(V_1 \times ... \times V_m)/\mathcal{S} = \left\{ \sum_{\substack{v \in V \\ \text{(finite)}}} \lambda_v v + \mathcal{S} \;\middle|\; \lambda_v \in R \right\}$$

and we denote the simple cosets via:

$$v_1 \otimes ... \otimes v_m := (v_1, ..., v_m) + \mathcal{S}$$

and we call them **simple tensors**. The more general elements, **tensors**, are just finite formal linear combinations of simple tensors.

[Exercise: Prove for example that $v \otimes (w + w') = v \otimes w + v \otimes w'$. Hint: You can add and subtract by elements of $\mathcal{S}$ maintaining the coset. Note that in the *direct product structure* $(v, w + w') \neq (v, w) \dot{+} (v, w') := (v + v, w + w')$. So what we have done is force isolated component linearity by making $(v, w + w') \equiv (v, w) + (v, w')$ $(mod\ S)$ in the new $+$.]

[Exercise: Prove: $\underbrace{V \otimes ... \otimes V}_{k} \otimes \underbrace{V^* \otimes ... \otimes V^*}_{l} \cong \mathcal{T}_l^k(V)$, for a vector space V via the map:
$v_1 \otimes ... \otimes v_k \otimes \omega_1 \otimes ... \otimes \omega_l \quad \mapsto \quad \varphi$, where
$\varphi(\omega_1', ..., \omega_k', v_1', ..., v_l') := v_1(\omega_1') \cdot ... \cdot v_k(\omega_k') \cdot \omega_1(v_1') \cdot ... \cdot \omega_l(v_l')$ and extend linearly.]

|<<|

References

[1] Axler, S. (2020). *Measure, Integration & Real Analysis.* Springer.

[2] Bishop, R. L., Goldberg, S. L. (1980). *Tensor Analysis On Manifolds.* Dover, New York.

[3] Dummit, D. S., Foote, R. M. (2004). *Abstract Algebra.* John Wiley & Sons Inc.

[4] Friedberg, S.H., Insel A. J., Spence, L. E. (2019). *Linear Algebra.* Pearson.

[5] Hatcher, A. (2002). *Algebraic Topology.* Cambridge University Press.

[6] Kay, D. C. (2011). *Tensor Calculus.* McGraw-Hill.

[7] Lee, J. (2013). *Introduction to Smooth Manifolds.* Springer, New York.

[8] Lee, J. (1997). *Riemannian Manifolds: An Introduction to Curvature.* Springer, New York.

[9] Miranda, R. (1997). *Algebraic Curves and Riemann Surfaces.* American Mathematical Society.

[10] Ross, K. A. (1980). *Elementary Analysis: The Theory of Calculus.* Springer.

[11] Boothby, W. M. (1975). *An Introduction to Differentiable Manifolds and Riemannian Geometry.* Academic Press.

[12] Spivak, M. (1965). *Calculus on Manifolds.* Perseus Books Publishing, L.L.C.

[13] Stewart, J. (2008). *Calculus: Early Transcendentals 6 Ed.* Thompson Brooks/Cole

[14] Wald, R. M. (1984). *General Relativity.* University of Chicago Press.

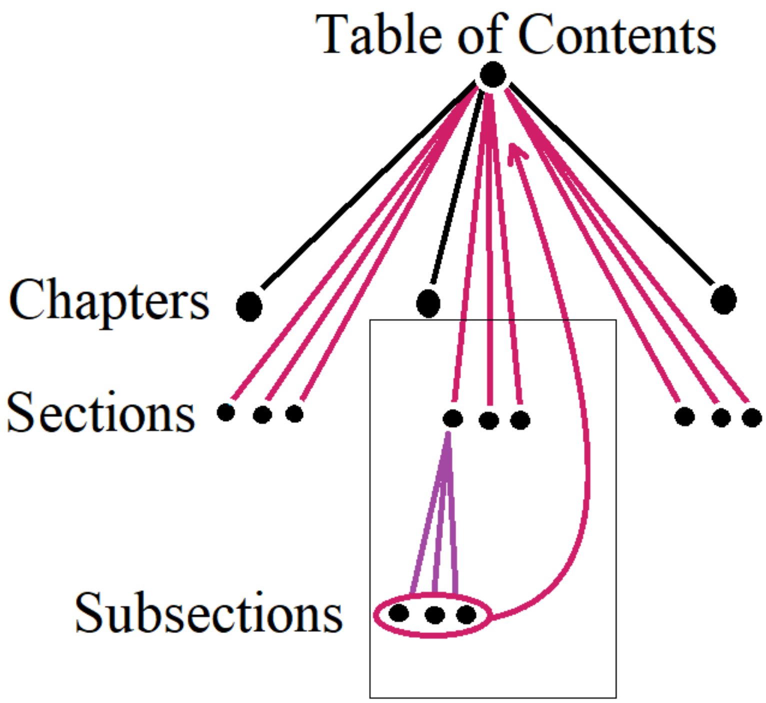
[15] Wikipedia contributors. Parallel transport. Wikipedia, The Free Encyclopedia. December 6, 2020, 14:22 UTC. Available at:

<< https://en.wikipedia.org/w/index.php?title=Parallel_transport&oldid=992672595. >>

Accessed December 30, 2020.

Using this PDF:

This text was designed with a hyperlink navigational structure indicated by the following diagram:



with downward arrows usually being labeled by name and any upward arrows being given by the “/<</” or “/</” icons in the bottom left corner of each page. There may also be internal links not specified here. Not all sections link to their subsections, the ones that do will be indicated with a ‘•’ symbol in the table of contents. If you keep this in mind while reading, it makes accessing information in the document easier. Press back to get started.