Category Theory Proper

By: Kevin Smith

Summer, 2020

>> Read Me

Table of Contents [Compressed]:

Table of Contents [Expanded]:

<< PART I >>

<< PART II >>

<< **PART III** >>

Introduction

Categories are generalizations of at least *algebraic structures* and *geometric structures* and *logical structures*. We study Category Theory Proper to get general results and a perspective on multiple subjects at once. In my experience, some attacking points from the concrete side are:

1.) The Fundamental Group of a Topological Space can be thought of as a *functor* (between categories **Top** and **Grp**) assigning a *connected space* $\boldsymbol{X}$ to its (non-point-dependent) fundamental group $\boldsymbol{G := \pi_1(X)}$. **[Algebraic Topology]**

2.) Subgroups of the Fundamental Group are in Galois-Correspondence with Sub-covering spaces of the Universal Covering Space $\mathscr{U} : \boldsymbol{U \to X}$ for the base space $\boldsymbol{X}$. Turns out that $\mathscr{U} : U \to X$ satisfies a *universal property* and is an example of an *initial object* in the *category* $\boldsymbol{TopCov(X)}$. **[Algebraic Topology]**

3.) In **[Riemann Surfaces]** we have sheaves (e.g. of meromorphic functions, differential forms, tensors, etc.) and cohomology groups given by chain complexes specified by such sheaves and particular topological covers for the Riemann surface. The (intrinsic, non-cover-dependent) cohomology groups are examples of direct limits (of systems of groups). These three notions generalize to category theory.

4.) In **[Modal Logic]**, we have the notion of duality appear between associated categories of $\boldsymbol{F}$-algebras and coalgebgras (that is: $\boldsymbol{Alg(F)}$ and $\boldsymbol{CoAlg(F)}$, where $\boldsymbol{F}$ is a functor built from the operations and identities of the algebraized modal logic).

† See reference projects [**?**, **?**, **?**] for more information on some backing theory presented in (2-4).

5.) In **[Representation Theory]**, we have an example of adjoint functors given by *Restriction* and *Induction* of representations, $\boldsymbol{\rho\downarrow^G_H, \rho\uparrow^G_H}$.

These are motivating points for studying the theory, but are not exclusively what we explore in this paper. This is as much a research paper for myself as it is an instructive tool for those who read it. The arrangement appeals to my abstract algebraic intuition.

This is by no means a comprehensive text and will be purposefully cut short as to give the reader an initial bar to be reached in studying. This bar is reinforced by the qualifying exam problems being solvable at the point of completion.

Some reference notes are given on the next page and a full bibliography is given in the back of the text.

I received my Masters in Pure Mathematics from California State University, Long Beach in the Spring of 2020; I did a year of Ph.D. study at ASU in 2018 (timeline correct!), where I studied category theory under Nancy Childress; Bachelor's from UC Irvine; Associates from Orange Coast College, where I was introduced to the subject in a summer seminar lead by Arthur Moore. I immediately was drawn to the subject upon first encounter and have had a difficult time leaving it alone! It was my pleasure to hold discourse with Rico Vincente (my collegue from CSULB) on some of the qual problems listed here. Thanks to my family for letting me rant this abstract nonsense to them!

Overall Section Citations:

I.1: [?],[?],[?],[?]
I.2: [?] and [?]

II.1: [?]
II.2: [?], [?], and [?]
II.3: [?] and [?]
II.4: [?]

III.1: HA University Comprehensive Exams.
III.2: [?], [?], and [?]

In terms of quick references, Wikipedia has great articles on the subject:

[Glossary of Terms] [Basic Concepts]

Otherwise, here are some more that you should go download. A complete list is given in the bibliography.

<< PART I >>

1. Universes and Categories

As described in [?] (p.16-22), we *expand* the [**Zermelo-Fraenkel Set Theory**] by introducing *universes*. [See (p.19 of [?]) for ZF-Set-Theory **Axioms I-VIII** discussion as well.]

- -

• Def: A **universe $\mathfrak{U}$** is a set (of sets) subject to the following *closure axioms*:

1.) $A \in \mathfrak{U} \implies A \subset \mathfrak{U}$,

[Set/Class Axiom]

2.) $A \in \mathfrak{U}$ and $B \in \mathfrak{U} \implies \{A, B\} \in \mathfrak{U}$ (set with the elements A, B),

[Primitive Union Closure]

3.) $A \in \mathfrak{U} \implies \mathscr{P}(A) \in \mathfrak{U}$

[Power Set Closure],

4.) If $J \in \mathfrak{U}$ and if $f : J \to \mathfrak{U}$ is a map, then $\left(\bigcup_{j \in J} f(j)\right) \in \mathfrak{U}$

[Arbitrary Union over Elements of the Universe Closure].

New ZFST Axiom: Every set is an element of a universe. Thus, in particular, every universe is an element of a *higher universe*. $\mathfrak{U} \in \mathfrak{V}$

• Def: With respect to a universe $\mathfrak{U}$. We define **sets** (more exactly: **$\mathfrak{U}$-sets**) to be the *elements* of $\mathfrak{U}$. **Classes** (more exactly: **$\mathfrak{U}$-classes**) are the *subsets* of $\mathfrak{U}$.

Sets are classes (by definition), but not all subsets of the universe are elements of the universe [Exercise: Find a counter-example], so there are classes which are not sets. We call things in the complement **proper classes**.

The axioms in this expansion of $ZFST$ were designed to imply that the usual constructions of set theory, carried out with elements of $\mathfrak{U}$, land back in $\mathfrak{U}$.

• Proposition: (p.17 [?])
i.) If $A \in \mathfrak{U}$, then every subset of A is also an element of $\mathfrak{U}$.

ii.) For any two sets A and $B \in \mathfrak{U}$, we also have $A \times B$ and $\{f : A \to B\}$ are in $\mathfrak{U}$.

iii.) The product set $\prod_{j \in J} A_j$ is an element of $\mathfrak{U}$ if the indexing set and all the family members are in $\mathfrak{U}$.

Proof: [Exercise.]

• Def: (pg.1 [?])
Let us assume a particular universe $\mathfrak{U}$ is given and define the following symbols:

$$\mathcal{C} := \left\{ Obj(\mathcal{C}),\ Hom(\mathcal{C}),\ \circ,\ Id(\mathcal{C}) \right\}$$

- -

$Obj(\mathcal{C}) := \{A, B, C, ..., X, Y, Z, ...\}$, we call the elements in this collection, **objects**.

- -

$Hom(\mathcal{C}) := \left\{ Hom_{\mathcal{C}}(A, B) \;\middle|\; A, B \in Obj(\mathcal{C}) \right\}$;

$$Hom_{\mathcal{C}}(A, B) := \left\{ f : A \to B \;\middle|\; f \text{ preserves the structure of objects in } \mathcal{C} \right\},$$

we call the elements $(f : A \to B)$, **morphisms** with *domain* A and *codomain* B.

- -

Composition of morphisms, is taken to be an *associative binary operation*:

$$\circ : Hom(C) \times Hom(C) \to Hom(C);$$

$$(f, g) \mapsto g \circ f$$

defined only when $dom(g) = codom(f)$. For instance, we also write

$$\circ : Hom(A, B) \times Hom(B, C) \to Hom(A, C),$$

for arbitrary triple of objects (A, B, C).

- -

$Id(\mathcal{C}) := \left\{ Id_A : A \to A \mid A \in Obj(\mathcal{C}) \right\} \subseteq Hom(\mathcal{C})$,

we call these **identity morphisms** and define them to be such that $\forall A, B, \forall f \in Hom(A, B)$,

$$f \circ Id_A := f \qquad \text{and} \qquad Id_B \circ f := f.$$

- -

The four pieces of data listed above define a **category**, $\mathcal{C}$.

(Continues)

Notes:
1.) When context is known, we drop the $\mathcal{C}$ in $Hom_{\mathcal{C}}(A, B)$.

2.) Alternative notations are used such as $|\mathcal{C}| := Obj(\mathcal{C})$, $Mor(\mathcal{C}) := Hom(\mathcal{C})$, or:

$$[A, B]_{\mathcal{C}} = Hom_{\mathcal{C}}(A, B) \qquad \text{or just} \qquad [A, B] = Hom(A, B).$$

3.) On (p.1 [?]), Schubert defines another axiom that says the $Hom(A, B)$ collections are disjoint for distinct pairs (A, B). But [?] does not list it. [Exercise: Why would we need this?].

4.) Schubert also mentions that identities uniquely determine objects and vice versa. So that

$$Obj(\mathcal{C}) \leftrightarrow Id(\mathcal{C}) \subseteq Hom(\mathcal{C}).$$

Small and Large Categories:

The following is based on p.18 in [?], p.11/23 [?] but there are discrepancies in each so we make our own sensible definition.

• Def: Given a universe $\mathfrak{U}$, a category is called $\mathfrak{U}$-small if $Hom(\mathcal{C})$ is a $\mathfrak{U}$-set.

Otherwise, a $\mathfrak{U}$-large category is one such that $Hom(\mathcal{C})$ is a (proper) $\mathfrak{U}$-class.

In general, we refer to $\mathfrak{U}$-categories as ones where $Hom(\mathcal{C})$ is a $\mathcal{U}$-class (could be a set or proper class, ambiguous).

In practice, one can ignore the size considerations given above and just refer to the $Obj(\mathcal{C})$ and $Hom(\mathcal{C})$ entities as "collections" instead of sets or classes to avoid that discussion until necessary. Moreover, the references simply consider a fixed universe containing the usual number systems $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ etc. for concreteness. Whether this is sufficient to cover all examples of categories we will encounter, I am not sure. The reader should care about universes until Section I.2 and then the take away should be the definition of a category given by $\mathcal{C} = \{Obj(\mathcal{C}), Hom(\mathcal{C}), \circ\}$ from then on out.

1.1 Abstract Examples of Categories:

Before listing familiar examples, we are going to describe special types of categories.

• <u>Def:</u> A **discrete category** is one such that $\boldsymbol{Id(\mathcal{C}) = Mor(\mathcal{C})}$.

• <u>Def:</u> Useful later in *Presentations of Functors* and separately in *Limits*, we will develop the examples of:

> ***Path Categories***
> ***Categories of Diagrams of Type*** $\boldsymbol{\Sigma/K}$, and
> ***Categories of Diagrams Schemes.***

[See Part II, Section 1].

These allow us to consider categories constructed from (or reduced to) underlying graphs (as in graph theory). The graphs are also referenced as **patterns**. These typify categories, allow us to describe functors graphically, and help us to distinguish different types of limits, which play a big role.

• <u>Def:</u> A **connected category** $\boldsymbol{\mathcal{C}}$ is one such that any two objects can be linked together by a sequence of Hom sets that are non-empty [**?**].

• <u>Def:</u> An **enriched category** is one in which all of the $\boldsymbol{Hom(A, B)}$ collections have extra structure, especially algebraic. This leads to the notions of *Sheaves*.

• <u>Def:</u> Given any category, $\boldsymbol{\mathcal{C}}$, the **dual category** denoted $\boldsymbol{\mathcal{C}^{op}}$ has the same object collection except with the morphisms reversed, however we maintain the distinction with the labels. That is: $\forall \boldsymbol{A, B \in Obj(\mathcal{C})}$, we have:

$$\boldsymbol{A^{op}, B^{op} \in Obj(\mathcal{C}^{op}),} \text{ are such that } \boldsymbol{A^{op} := A} \text{ and } \boldsymbol{B^{op} := B} \text{ and}$$

$$\boldsymbol{Hom_{\mathcal{C}^{op}}(A^{op}, B^{op}) := Hom_{\mathcal{C}}(B, A)}, \text{ i.e. } \boldsymbol{(f^{op} : A^{op} \to B^{op}) := (f : B \to A)}$$

Of course composition is defined as $\boldsymbol{f^{op} \,\dot{\circ}\, g^{op} := g \circ f}$. [**?**]

This notion of dual allows us to treat only covariant functors (when we get to it), by pre-composing with the ***duality functor****:* $\boldsymbol{Op : \mathcal{C} \to \mathcal{C}^{op}}$. [Exercise: Prove that pre-composing with Op changes contra- to co- and post-composing changes a co- to contra-. See the section on functors first of course.]

Abstract Examples of Categories (Continued):

• Def: Suppose for a given category $\mathcal{C}$, the elements of $Hom(\mathcal{C})$ are called **1-morphisms**. If for any pair of morphisms $f, g \in Hom(A, B)$ we have the enrichment $Hom(f, g)$ of **2-morphisms** (also called commutative squares), we refer to $\mathcal{C}$ as a **2-category**. Further enrichment of the 2-morphisms leads to **n-morphisms** and **n-categories**. The study of n-categories (*strict* and *weak*) is known as **higher category theory** [For more on n-categories, see [?]].

Example: A particular example of a 2-category, $\widetilde{\mathcal{C}}$, is the *category of categories*, where:

$$\{Obj(\widetilde{\mathcal{C}}), Hom(\widetilde{\mathcal{C}})\} := \Big\{\{categories\ \mathcal{C}\}, \{functors\ F : \mathcal{C} \to \mathcal{D}\}_{\mathcal{C},\mathcal{D}}\Big\}.$$

Natural transformations $\eta : F \to G$ (to be defined) provide the 2-morphisms.

Recall the discussion on universes and small versus large categories. We introduce the following abstract examples of categories (see p.19-22 in [?]):

$Ens \equiv Set$: The category of $\mathfrak{U}$-sets and their maps.
ENS: The category of $\mathfrak{V}$-sets and their maps. ($\mathfrak{U} \in \mathfrak{V}$)

cat: The category of $\mathfrak{U}$-small categories and functors between such categories.
Cat: The category of all $\mathfrak{U}$-categories (small and large) and functors between such categories.
CAT: The category of $\mathfrak{V}$-small categories and like functors. ($\mathfrak{U} \in \mathfrak{V}$)

$[\mathcal{C}, \mathcal{D}]_{\mathcal{X}} \equiv Fct(\mathcal{C}, \mathcal{D})_{\mathcal{X}}$: Category of functors between categories $\mathcal{C}$ and $\mathcal{D}$ with associated natural transformations. Note: The functor category's size depends on $\mathcal{X} = cat/Cat/CAT$, since $\mathcal{C}, \mathcal{D} \in Obj(\mathcal{X})$.

$Nat(\mathcal{C}, \mathcal{D})_{\mathcal{X}}$ also exists (a level 3-category between $\mathcal{C}$ and $\mathcal{D}$).

In terms of containment, we have:

$$Ens \subset ENS \qquad \text{and} \qquad cat \subset Cat \subset CAT \qquad \text{and} \qquad [\mathcal{C}, \mathcal{D}]_{cat} \subset [\mathcal{C}, \mathcal{D}]_{Cat} \subset [\mathcal{C}, \mathcal{D}]_{CAT}$$

[Exercise: Prove these inclusions.]

1.2 Concrete Examples of Categories:

Now, let's list some familiar examples of basic categories (reader should pick one and show that the definition is satisfied). We won't be too rigorous here.

$\boldsymbol{Rel}$ = {$\mathfrak{U}$-sets, binary relations between them},

$\boldsymbol{Grp}$ = {groups, group homomorphisms},
$\boldsymbol{Ring}$ = {rings, ring homs},
$\boldsymbol{Field}$ = {fields, field homs},
$_{\boldsymbol{R}}\boldsymbol{Mod}$ = {left $\boldsymbol{R}$-modules, $\boldsymbol{R}$-linear maps}
$\boldsymbol{Vec_K}$ = {vector spaces over a field $\boldsymbol{K}$, $\boldsymbol{K}$-linear maps},
$\boldsymbol{Alg(F)}$ = {$\boldsymbol{F}$-algebras $\boldsymbol{(A, \alpha : F(A) \to A)}$ together with appropriate morphisms},
$\boldsymbol{CoAlg(F)}$ = {$\boldsymbol{F}$-coalgebras $\boldsymbol{(A, \alpha : A \to F(A))}$ together with appropriate morphisms},

$\boldsymbol{Top}$ = {topological spaces, continuous functions},
$\boldsymbol{Pos(X)}$ = {open sets in the topology on $\boldsymbol{X}$ and morphisms are given by the partial-order $\leq$},
$\boldsymbol{TopCov(X)}$ = {topological coverings of $\boldsymbol{X}$ together with covering morphisms},

$\boldsymbol{Man^r}$ = {smooth manifolds, $\boldsymbol{r}$-differentiable maps},
$\boldsymbol{RS}$ = {Riemann surfaces $\mathscr{M}$, holomorphic maps},

Many more examples exist and are created for different theories (see [?] (p.2)).

2. Functors and Natural Transformations

• Def: A **(covariant) functor** between two categories is a **bimap** between both the collections of objects and morphisms, obeying certain "structure preservation" properties.

Particularly, we write $\boldsymbol{F : \mathcal{C} \to \mathcal{D}}$ and mean $\boldsymbol{F : \{Obj(\mathcal{C}), Hom(\mathcal{C})\} \mapsto \{Obj(\mathcal{D}), Hom(\mathcal{D})\}}$, where:

1.) $\forall \boldsymbol{X} \in \boldsymbol{Obj(\mathcal{C})}, \qquad \boldsymbol{F(X)} \in \boldsymbol{Obj(\mathcal{D})}$,

2.) $\forall \boldsymbol{f} \in \boldsymbol{Hom_{\mathcal{C}}(X,Y)}, \qquad \boldsymbol{F(f)} \in \boldsymbol{Hom_{\mathcal{D}}(F(X),F(Y))}$

Such that:

3.) $\forall \boldsymbol{f,g} \in \boldsymbol{Hom(A,B), Hom(B,C)}$ respectively, we have $\boldsymbol{F(g \circ f) = F(g) \circ F(f)}$

4.) $\forall \boldsymbol{X} \in \boldsymbol{Obj(\mathcal{C})}$, $\boldsymbol{F(Id_X) = Id_{F(X)}}$.

That is, composition and identities are preserved. Note $\boldsymbol{F =: (F_1, F_2)}$ may be used.

• Def: A **(contravariant) functor**, $\boldsymbol{F : \mathcal{C} \to \mathcal{D}}$, is identical to the above except in items (2) and (3) we have:

2.') $\forall \boldsymbol{f} \in \boldsymbol{Hom_{\mathcal{C}}(X,Y)}, \qquad \boldsymbol{F(f)} \in \boldsymbol{Hom_{\mathcal{D}}(F(Y),F(X))}$

3.') $\forall \boldsymbol{f,g} \in \boldsymbol{Hom(A,B), Hom(B,C)}$ respectively, we have $\boldsymbol{F(g \circ f) = F(f) \circ F(g)}$.

Contravariant functors are said to be "arrow reversing". It should also be noted that:

$$\text{(contra.)}\ \boldsymbol{F : \mathcal{C} \to \mathcal{D}} \qquad = \qquad \text{(cov.)}\ \boldsymbol{F : \mathcal{C}^{op} \to \mathcal{D}}$$

or equivalently

$$\text{(contra.)}\ \boldsymbol{F : \mathcal{C}^{op} \to \mathcal{D}} \qquad = \qquad \text{(cov.)}\ \boldsymbol{F : \mathcal{C} \to \mathcal{D}}$$

[Exercise: Prove these equalities. Recall our discussion of $\boldsymbol{F \circ Op}$ and $\boldsymbol{Op \circ F}$ from Section I.1.1].

• Def: We say a **multi-functor** is one whose source is a *product category* (see Section II.4) and who's image preserves composition in a way specified by the variance:

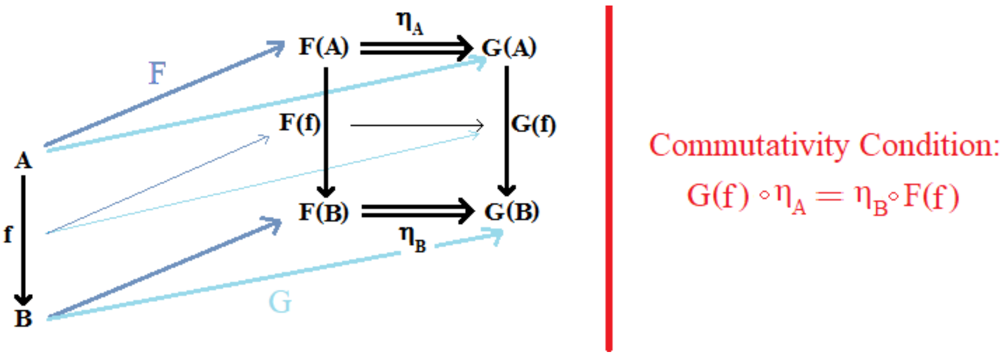
E.g. for a **(co-co)-bifunctor**, $\boldsymbol{F\big((f,g) \circ (f',g')\big) = F(f,g) \circ F(f',g')}$.

E.g. for a **(contra-co)-bifunctor**, $\boldsymbol{G\big((f,g) \circ (f',g')\big) = G(f',g) \circ G(f,g')}$.

A functor is said to have **mixed variance** if its *partial functors* exhibit separately co- and contra- variance. By **partial functor**, we mean ones where all variables are fixed except for the i^{th} one.

The particulars of the variance can be listed in a vector such as parts contra- and co- $(2,3)$ or more explicitly $(1,0,0,1,0)$, where 1 is contra-, 0 is co-variant.

• Def: A **natural transformation**, $\eta : F \to G$, between functors $F, G : \mathcal{C} \to \mathcal{D}$, is an *indexed collection* of maps (also called **components**), $\{\eta_A\}_{A \in Obj(\mathcal{C})}$ making the following diagrams commute for arbitrary $f : A \to B$:



That is, this family transforms one functor into another by altering the images subject to the commutativity conditions (red).

• Def: A **natural isomorphism** between functors is a natural transformation η in which all the components are *isomorphisms* as defined by the category. (More on types of morphisms later in Part II, Section 2). We denote naturally isomorphic functors via $F \cong G$.

• Def: The term **natural equivalence** is also used. Two categories are said to be **equivalent** if there exist two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that both compositions:

$$F \circ G \cong Id_{\mathcal{D}} \text{ and } G \circ F \cong Id_{\mathcal{C}}$$

are naturally isomorphic to the *identity functors*. This is weaker than **isomorphic categories** in which we have equality for the compositions and identities (p.21 [?]).

2.1 Primitive Notions Regarding Functors:

• Def: (p.25 [?]) A functor $\boldsymbol{F : \mathcal{C} \to \mathcal{D}}$ is said to be **faithful** (resp. **full**, **fully faithful**) if:

$$\forall A, B \in obj(\mathcal{C}) \qquad F_{AB} : Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{D}}(F(A), F(B))$$

is *injective* (resp. *surjective*, *bijective*).

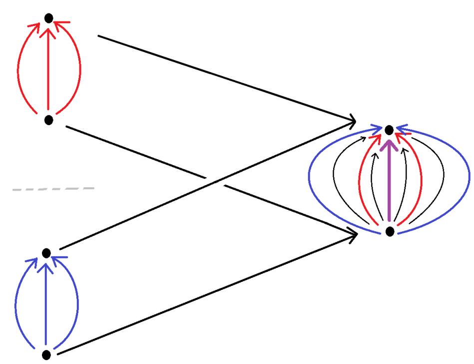
• Def: $\boldsymbol{F : \mathcal{C} \to \mathcal{D}}$ is called an **embedding of $\mathcal{C}$ into $\mathcal{D}$** if:

$$F_2 : Hom(\mathcal{C}) \to Hom(\mathcal{D})$$

is injective.

- -

Notes:
1.) Faithful functors need not be injective (i.e. need not be embeddings), as the following illustrates:



2.) Embeddings yield *subcategories* in the image, whereas faithful functors do not (see p.25 [?] for a counterexample). Full functors also form a subcategory in the image.

[Exercise: Prove the statements in (2) with subcategory definition in Section II.4.1]

• Def: (p.16 [?]) $\boldsymbol{F : \mathcal{C} \to \mathcal{D}}$ is called **essentially surjective** if $\forall \boldsymbol{Y} \in \boldsymbol{obj(\mathcal{D})}$ there exists $\boldsymbol{X} \in \boldsymbol{obj(\mathcal{C})}$ such that $\boldsymbol{F(X) \cong Y}$ (i.e. the two objects are *isomorphic*).

• Def: The **Identity Functor**, $\boldsymbol{Id_{\mathcal{C}}}$, mentioned previously is just the bi-identity map on the objects and morphisms of $\mathcal{C}$.

(Continues)

• Def: Given two categories $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{D}}$, and $\boldsymbol{Y_0} \in \boldsymbol{obj(\mathcal{D})}$, we define the **constant functor**:

$$\boldsymbol{\Delta_{Y_0} : \mathcal{C} \to \mathcal{D}};$$

$$\forall \boldsymbol{X} \in \boldsymbol{Obj(\mathcal{C}),\ X \mapsto Y_0}, \text{ and}$$

$$\forall \boldsymbol{A, B}, \forall \boldsymbol{f} \in \boldsymbol{Hom(A, B),\ f \mapsto Id_{Y_0}}.$$

Note: Compare this later with the constant diagrams $\boldsymbol{A_\Sigma : \Sigma \to A}$ for some object $\boldsymbol{A} \in \boldsymbol{\mathcal{C}}$ and diagram $D : \boldsymbol{\Sigma \to \mathcal{C}}$ (see Section II.2).

• Def: We wait until Part II, Section 4 to define products and coproducts of categories, but assume we may speak of $\boldsymbol{\mathcal{C} \times \mathcal{D}}$ or $\boldsymbol{\mathcal{C} \amalg \mathcal{D}}$. Then we have the associated **Projection** and **Injection Functors** extrapolated from their set-theoretic counterparts.
For example: $\boldsymbol{\mathcal{C} \times \mathcal{D} \to \mathcal{C}}$ or $\boldsymbol{\mathcal{D} \to \mathcal{C} \amalg \mathcal{D}}$.

• Def: Given a category whose objects contain extra structure than just being sets (Groups for example). The **Forgetful Functor** maps every object to its underlying set. By extension, the morphisms become just set maps (we forget that they preserve structure).

Symbolically:

$$\boldsymbol{F : \mathcal{C} \to Set}$$

$$\boldsymbol{X \mapsto |X|} \qquad \text{and} \qquad \boldsymbol{(f : X \to Y) \mapsto (F(f) : |X| \to |Y|)}$$

Similarly, we may forget only part of a structure (p.7 [?]). For example algebraic structures can have multiple operations.

• Def/Prop: (p.8 [?]) The **Power Set Functor** is a contravariant functor given by:

$$\boldsymbol{\mathscr{P} : Set \to Set};$$

$$\boldsymbol{A \mapsto \mathscr{P}(A)}, \text{ and}$$

$$\boldsymbol{(f : A \to B) \mapsto (P(f) : \mathscr{P}(B) \to \mathscr{P}(A))};$$

$$\boldsymbol{X \mapsto f^{-1}(X)}.$$

2.2 Some More Notions and Examples of Functors:

Each of the following examples takes a bit to explain and hence warrants its own navigation menu. Each particular example illuminates some definitions in category theory as well.

Sub-Subsection Contents:

>> Jump to Part II

1: Hom Functors (Co-, Contra-, and Contra-Co-):

• Def: (pp.7-8 from [?]) Suppose $\boldsymbol{A, Z \in obj(\mathcal{C})}$ and $\boldsymbol{f \in Hom(X, Y)}$, with $\boldsymbol{A}$ fixed.

Then we define the **co-variant Hom-functor** $\boldsymbol{H^A : \mathcal{C} \to Sets}$ via:

$$\boldsymbol{H^A(Z) := Hom(A, Z)} \qquad \text{and}$$

$$\boldsymbol{H^A(f) : Hom(A, X) \to Hom(A, Y)},$$

where for $\boldsymbol{u \in Hom(A, X)}$ we define $\boldsymbol{[H^A(f)](u) := f \circ u}$.

- -

• Def: We define the **contra-variant Hom-functor** $\boldsymbol{H_A : \mathcal{C} \to Sets}$ via:

$$\boldsymbol{H_A(Z) := Hom(Z, A)} \qquad \text{and}$$

$$\boldsymbol{H_A(f) : Hom(Y, A) \to Hom(X, A)},$$

where this time, for $\boldsymbol{u \in Hom(Y, A)}$ we define $\boldsymbol{[H_A(f)](u) := u \circ f}$.

- -

[Exercise: Prove each type of variance claimed in the definition.] >> [Proof]

- -

• Def: (p.11 [?]) Let $\boldsymbol{X, X', Y, Y' \in obj(\mathcal{C})}$, $\boldsymbol{f \in Hom(X, X')}$, and $\boldsymbol{g \in Hom(Y, Y')}$.
Then the *contra-co-variant* **Hom-functor** $\boldsymbol{H : \mathcal{C} \times \mathcal{C} \to Sets}$ is defined via:

$$\boldsymbol{H((X, Y)) := Hom(X, Y)} \qquad \text{and}$$

$$\boldsymbol{H((f, g)) : Hom(X', Y) \to Hom(X, Y')},$$

where for $\boldsymbol{u \in Hom(X', Y)}$ we define $\boldsymbol{[H((f, g))](u) := g \circ u \circ f}$.

Notes:
1.) We can retrieve $\boldsymbol{H^A}$ by fixing the second object variable $\boldsymbol{Y := A}$ and setting $\boldsymbol{g = 1_A}$. Similarly we can retrieve $\boldsymbol{H_A}$.
2.) The notation gets disgusting. We can also denote $\boldsymbol{H^A(Z)}$ by $[\boldsymbol{A, Z}]$ and $\boldsymbol{H^A(f)}$ by $[\boldsymbol{A, f}]$. This leads to denoting $\boldsymbol{H((f, g))}$ by $[\boldsymbol{f, g}]$ etc.
We also see $\boldsymbol{H(?, ??)}$ denoted by $[\boldsymbol{Op?, ??}]$ (p.12 [?]), where $\boldsymbol{Op : \mathcal{C} \to \mathcal{C}^{op}}$ is the **duality functor**.
3.) Special Note: We'll see $\boldsymbol{\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}}$ denoting a contra-co- bifunctor in doubly-covariant symbology as a shorthand. This quickly tells the reader which component is contra- etc.

2: Representable Functors, Universal Elements, and Yoneda's Map:

The following is based on p.25-28 [?].

• <u>Def:</u> A (covariant) functor $\boldsymbol{F : \mathcal{C} \to Set}$ is called **representable** if F is *naturally isomorphic* to a co-variant Hom-functor $\boldsymbol{H^A : \mathcal{C} \to Set}$.

Given an object $\boldsymbol{A}$, a particular **representation of $\boldsymbol{F}$** is then a particular natural isomorphism:

$$\eta^A : H^A \to F$$

$$\eta^A := \left\{ \eta^A_X : Hom(A, X) \xrightarrow{\cong} F(X) \right\}_{X \in obj(\mathcal{C})}$$

and we refer to the **representing object $\boldsymbol{A}$ for $\boldsymbol{F}$**.

• <u>Def:</u> Alternatively, a **representation of $\boldsymbol{F : \mathcal{C} \to Set}$** may be described by a pair:

$$(A, u), \text{ where } A \in Obj(\mathcal{C}) \text{ and } u \in F(A).$$

We call $\boldsymbol{A}$ the **representing object** still, but we now call $\boldsymbol{u}$ the **universal element** of the representation.

Although $\boldsymbol{A}$ is predetermined by $\boldsymbol{F}$ being representable, we have choice in which $\boldsymbol{u}$ we want. This description arises from the following.

• <u>Prop:</u> For $\boldsymbol{A} \in \boldsymbol{Obj(\mathcal{C})}$ and a given covariant functor $\boldsymbol{F : \mathcal{C} \to Set}$,

$$Nat(H^A, F) \leftrightarrow F(A),$$

that is, they are in 1-1 correspondence as sets.

- -

<u>Proof:</u> Next Page.

Proof (Smith): Let us define **Yoneda's map**:

$$\boldsymbol{Y : Nat(H^A, F) \to F(A)}$$

$$\boldsymbol{\eta^A \mapsto \eta^A_A(Id_A)}.$$

WTS the map $\boldsymbol{Y}$ is bijective.

- -

Accordingly, suppose we have two $\boldsymbol{\alpha^A, \beta^A} \in \boldsymbol{Nat(H^A, F)}$ such that $\boldsymbol{Y(\alpha^A) = Y(\beta^A)}$, then $\boldsymbol{\alpha^A_A(Id_A) = \beta^A_A(Id_A)}$.

Chasing down the explicit forms of the commutativity conditions for arbitrary object $\boldsymbol{X}$ and $\boldsymbol{f} \in \boldsymbol{H^A(X)}$ (by definition of natural transformations and restricting to $\boldsymbol{Id_A} \in \boldsymbol{H^A(A)}$) yields:

$$\boldsymbol{\alpha^A_X(f) = F(f)\alpha^A_A(Id_A)} \ ; \qquad \boldsymbol{F(f)\beta^A_A(Id_A) = \beta^A_X(f)}.$$

Together with the first equality, we have then $\boldsymbol{\alpha^A_X(f) = \beta^A_X(f)}$. Arbitrariness of $\boldsymbol{f}$ tells us $\boldsymbol{\alpha^A_X \equiv \beta^A_X}$ and then arbitrariness of $\boldsymbol{X}$ yields $\boldsymbol{\alpha^A = \beta^A}$. Hence $\boldsymbol{Y}$ is injective.

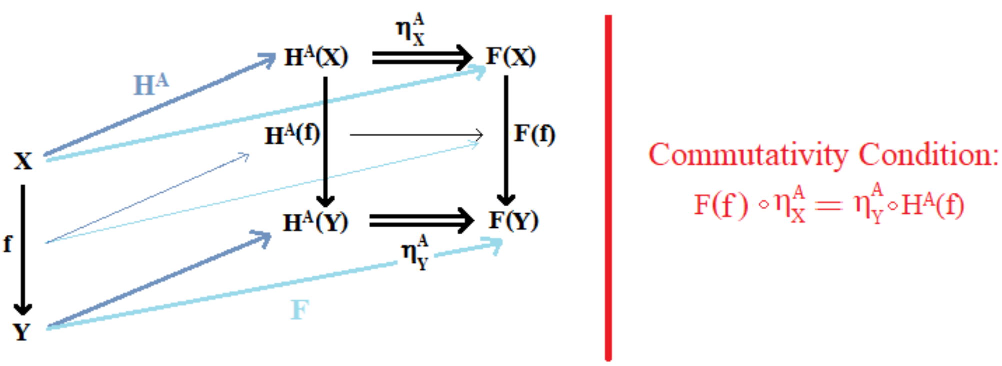
- -

Let $\boldsymbol{u} \in \boldsymbol{F(A)}$ be given. WTS $\exists\ \boldsymbol{\eta^A} \in \boldsymbol{Nat(H^A, F)}$ such that $\boldsymbol{Y(\eta^A) = u}$.

Accordingly, for arbitrary $\boldsymbol{X}$ and $\boldsymbol{g} \in \boldsymbol{H^A(X)}$ define:

$$\boldsymbol{\eta^A_X(g) := [F(g)](u)}$$

We need to show this yields the commutativity conditions for a natural transformation. Let's draw the usual picture:



We have then $\boldsymbol{F(f) \circ \eta^A_X(g) := F(f) \circ [F(g)](u) = [F(f \circ g)](u)}$ (by covariance).
And $\boldsymbol{\eta^A_Y([H^A(f)](g)) := \eta^A_Y(f \circ g) := [F(f \circ g)](u)}$. So by arbitrariness of $\boldsymbol{g} \in \boldsymbol{H^A(X)}$, the commutativity conditions are satisfied and $\boldsymbol{\eta^A}$ is a natural transformation.

Now, $\boldsymbol{Y(\eta^A) := \eta^A_A(Id_A) := [F(Id_A)](u) := Id_{F(A)}(u) = u}$ (by functor axiom). ■

3: Adjoint Functors:

Recall previously we defined the contra-co Hom functors.

• Def:(p.173 [?]) Let $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ be (covariant) functors. (F, G) is called a pair of **adjoint functors** if there is a natural isomorphism:

$$\left\{ \eta_{AB} : Hom_{\mathcal{C}}(F(A), B) \xrightarrow{\cong} Hom_{\mathcal{D}}(A, G(B)) \right\}_{(A,B) \in |\mathcal{D}| \times |\mathcal{C}|}$$

of *contra-co-variant* bifunctors from $\mathcal{D} \times \mathcal{C} \to Sets$.

In this case, G is called **right adjoint** to F (by means of η), F is called **left adjoint** to G and η is called an **adjunction isomorphism** for (F, G). We also say that $(\eta, F, G, \mathcal{C}, \mathcal{D})$ is an **adjunction system**.

[Exercise: Prove that *restriction* and *induction* (of *group representations* $\rho : G \to Aut(V)$ with $H \leq G$) yields an adjoint pair.]

4: Presentations of Functors with Diagrams:

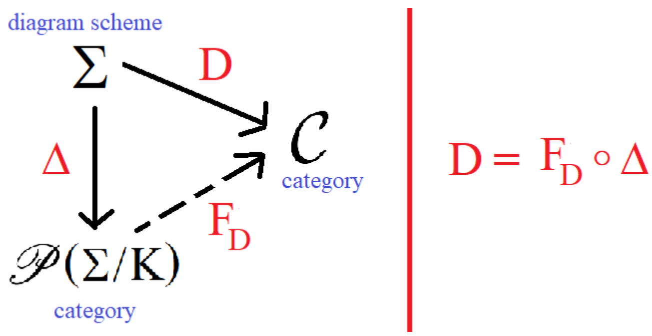
See Part II, Section 1 before proceeding in this sub-subsection.

• **Prop 6.3.2 from (p.40 [?]):**

Let $\boldsymbol{\Sigma}$ be a *diagram scheme* and let $\boldsymbol{K}$ be a set of *commutativity conditions* for Σ. There exists a (small) category, denoted $\mathscr{P}(\boldsymbol{\Sigma}/\boldsymbol{K})$, called the ***path category* belonging to $\boldsymbol{\Sigma}$ and $\boldsymbol{K}$**, together with a *diagram* $\boldsymbol{\Delta} : \boldsymbol{\Sigma} \to \mathscr{P}(\boldsymbol{\Sigma}/\boldsymbol{K})$ satisfying the following *universal property*:

If $\mathcal{C}$ is any category, then:

(i) If $\boldsymbol{D} : \boldsymbol{\Sigma} \to \boldsymbol{\mathcal{C}}$ is a diagram of type $\boldsymbol{\Sigma}/\boldsymbol{K}$, then $\exists!$ functor $\boldsymbol{F_D} : \mathscr{P}(\boldsymbol{\Sigma}/\boldsymbol{K}) \to \boldsymbol{\mathcal{C}}$ such that:



(ii) Moreover, there is an *isomorphism* of categories:

$$[\boldsymbol{\Sigma}/\boldsymbol{K}, \boldsymbol{\mathcal{C}}] \xrightarrow{\cong} [\mathscr{P}(\boldsymbol{\Sigma}/\boldsymbol{K}), \boldsymbol{\mathcal{C}}],$$

where the map for objects is given by the rule $\boldsymbol{D} \mapsto \boldsymbol{F_D}$.
[Exercise: Give the associated map on morphisms].

- -

Proof of Prop 6.3.2 (See [?]). ■

- -

Notes: This proposition tells us that any diagram factors through its diagram scheme's associated path category, uniquely giving rise to a functor F_D. On the other hand, one may consider a functor $\boldsymbol{F} : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{C}'}$ as a diagram if $\mathcal{C}$ is *small* with respect to its universe (recall diagram schemes were sets of vertices and arrows) and if we take the **underlying diagram scheme** of $\mathcal{C}$ via:

$$\boldsymbol{Ve := Obj(\mathcal{C})} \qquad \text{and} \qquad \boldsymbol{Ar := Mor(\mathcal{C})}$$

and the origin and end maps given by *domain* and *codomain* of the morphisms.

Hence we may write $\boldsymbol{D_F} : \boldsymbol{\Sigma_{\mathcal{C}}} \to \boldsymbol{\mathcal{C}'}$.

<< PART II >>

1. Regarding Diagram Schemes and Diagrams

1.1 Getting to Commutativity Conditions:

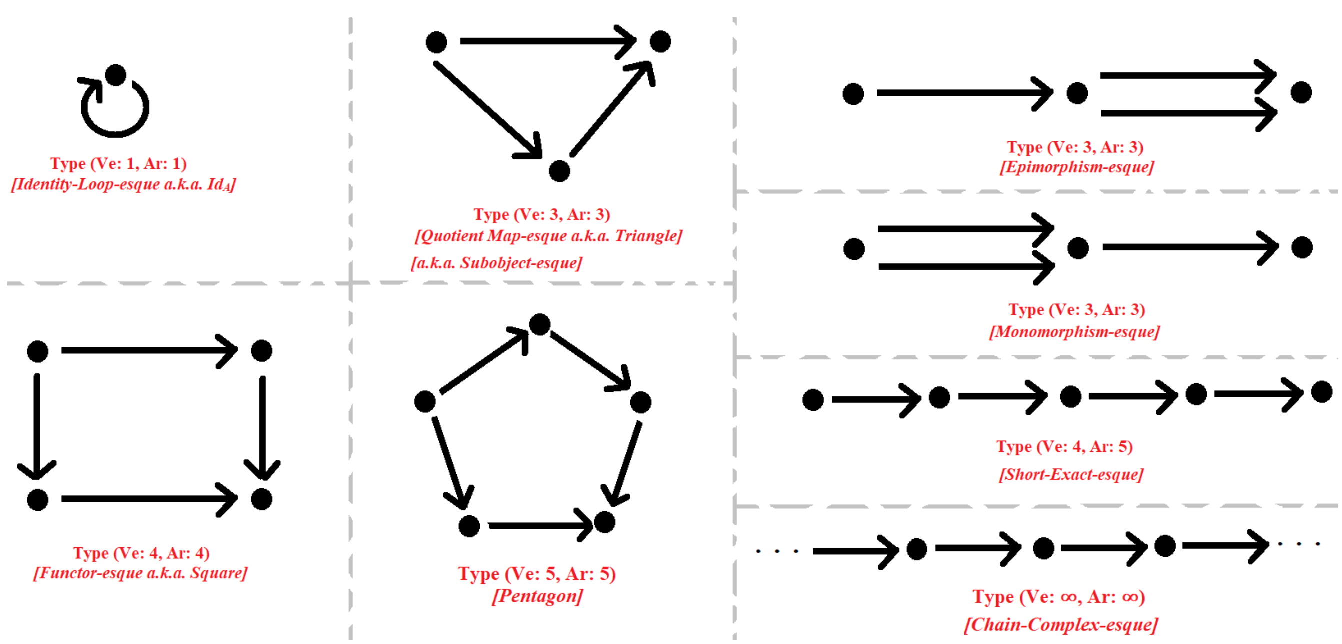
• <u>Def:</u> (p.37 [?]) A **diagram scheme $\boldsymbol{\Sigma}$** consists of two *sets*, called **vertices** and **arrows**, together with two maps called **origin** and **end**, which keep track of the *relational structure* in $\boldsymbol{Ve} \times \boldsymbol{Ar} \times \boldsymbol{Ve}$.

Syntactically, we may list diagram schemes as collections:

$$\boldsymbol{\Sigma} = \Big\{ \boldsymbol{Ve},\ \ \boldsymbol{Ar},\ \ \{\boldsymbol{o}, \boldsymbol{e} : \boldsymbol{Ar} \to \boldsymbol{Ve}\} \Big\}$$

such that arrows $\boldsymbol{a} \in \boldsymbol{Ar}$ are designated by $\boldsymbol{o}(\boldsymbol{a}), \boldsymbol{e}(\boldsymbol{a}) \in \boldsymbol{Ve}$.

*<u>Note:</u> As the author describes, diagram schemes are simply **oriented graphs**, but we phrase them in category language for our needs in the sequel. Below we have some examples of diagram schemes listed pictorially (one need not assign weights). We can of course apply set-theoretic constructions to these and others to get nasty pictures with high cardinality.*



• Def: (p.37) Let $\boldsymbol{\Sigma}$ be a diagram scheme and $\boldsymbol{\mathcal{C}}$ be an arbitrary category.
A **diagram in $\boldsymbol{\mathcal{C}}$ of *type* $\boldsymbol{\Sigma}$** is a map:

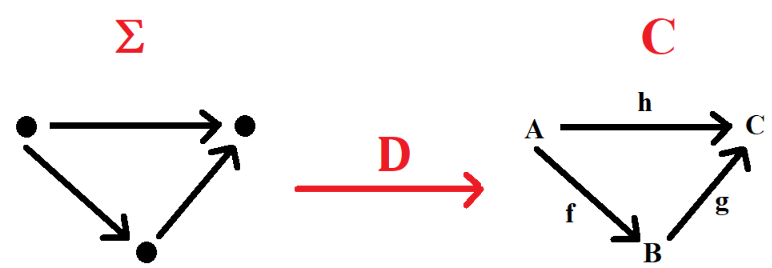
$$\boldsymbol{D : \Sigma \to \mathcal{C}}$$

such that:

i.) $\forall \boldsymbol{v} \in \boldsymbol{Ve},\ \boldsymbol{D(v)} \in \boldsymbol{Obj(\mathcal{C})}$ and
ii.) $\forall \boldsymbol{a} \in \boldsymbol{Ar},\ \boldsymbol{D(a)} \in \boldsymbol{Hom_{\mathcal{C}}\big(D(o(a)), D(e(a))\big)}$

That is, diagrams map vertices to objects and arrows to morphisms in a way compatible with origin and end.

- -



An example of a diagram of type $\boldsymbol{\Sigma}$ in $\boldsymbol{\mathcal{C}}$. Recall also the *pattern* terminology we used before.

(p.38 [?])
• Def: A **path in $\boldsymbol{\Sigma}$** (of length $\boldsymbol{n}$) is a sequence of arrows, $\boldsymbol{f = a_1 a_2 \ldots a_n}$ such that any two consecutive arrows in the list obey $\boldsymbol{o(a_{i+1}) = e(a_i)}$ and we upgrade the *origin* and *end* maps to paths by declaring:

$$\boldsymbol{o(f) := o(a_1)} \qquad \text{and} \qquad \boldsymbol{e(f) := e(a_n)}$$

• Def: Given two paths $\boldsymbol{f, g}$ in Σ with $\boldsymbol{o(g) = e(f)}$, we define their **path-product** $\boldsymbol{f * g}$ by the concatenated sequence hence $\boldsymbol{o(f * g) = o(f)}$ and $\boldsymbol{e(f * g) = e(g)}$.

- -

[Exercise: Prove *associativity* of $*$ for combine-able paths $\boldsymbol{f, g, h}$.]

[Exercise: Let $\boldsymbol{f, g \in Ar}$ such that $\boldsymbol{o(g) = e(f)}$ and let $\boldsymbol{D : \Sigma \to \mathcal{C}}$ be a diagram. Show that $\boldsymbol{D(f * g) = D(g) \circ D(f)}$ as morphisms in $\boldsymbol{\mathcal{C}}$ from $\boldsymbol{D(o(f))}$ to $\boldsymbol{D(e(g))}$. So that diagrams are compatible with path product.]

• Def: (p.39 [?]) If $\boldsymbol{\Sigma}$ is a diagram scheme, we construct its **trivial extension** $\boldsymbol{\Sigma_0}$ by adding for each $\boldsymbol{v \in Ve}$ an arrow $\boldsymbol{Id_v \in Ar}$ such that $\boldsymbol{o(Id_v) = v = e(Id_v)}$.

• <u>Def:</u> (p.39 [?]) A **commutativity condition** for a diagram scheme $\boldsymbol{\Sigma}$ is a pair of paths $(\boldsymbol{f}, \boldsymbol{g})$ in the *trivial extension* $\boldsymbol{\Sigma_0}$ of $\boldsymbol{\Sigma}$, where $\boldsymbol{f}$ and $\boldsymbol{g}$ have the *same origin and end*, that is:

$$\boldsymbol{o(f) = o(g)} \qquad \text{and} \qquad \boldsymbol{e(f) = e(g)}.$$

A diagram $\boldsymbol{D : \Sigma \to \mathcal{C}}$ is said to **satisfy the commutativity condition** $(\boldsymbol{f}, \boldsymbol{g})$ if we have:

$$\boldsymbol{D_0(f) = D_0(g)} \text{ as morphisms in } \boldsymbol{Hom_{\mathcal{C}}\big(D(o(f)), D(e(g))\big)}.$$

More generally, a diagram $\boldsymbol{D : \Sigma \to \mathcal{C}}$ is called **commutative** if it satisfies every possible commutativity condition.

• <u>Def:</u>(p.40 [?]) Let $\boldsymbol{\Sigma}$ be a diagram scheme and let $\boldsymbol{K}$ denote a (possibly empty) set of *commutativity conditions* for $\boldsymbol{\Sigma}$. A **diagram of type** $\boldsymbol{\Sigma/K}$ is a diagram $\boldsymbol{D_0 : \Sigma_0 \to \mathcal{C}}$ satisfying all commutativity conditions in $\boldsymbol{K}$. Which means the image of the diagram is subject to identifications.

Now we have seen types of diagrams in categories subject to commutativity conditions. Let's construct some other categories.

(Next Page)

1.2 Some Technical Examples of Categories Lying Around:

• Def: (p.41 [?]) Given a diagram scheme $\boldsymbol{\Sigma}$, we define the **free category over $\boldsymbol{\Sigma}$** or the **path category over $\boldsymbol{\Sigma}$**, denoted $\mathscr{P}(\boldsymbol{\Sigma})$, to be such that:

$$\boldsymbol{Obj(\mathscr{P}(\Sigma)) := Ve} \quad \text{and} \quad \boldsymbol{Hom_{\mathscr{P}(\Sigma)}(v,w)} := \{\text{paths from v to w}\}$$

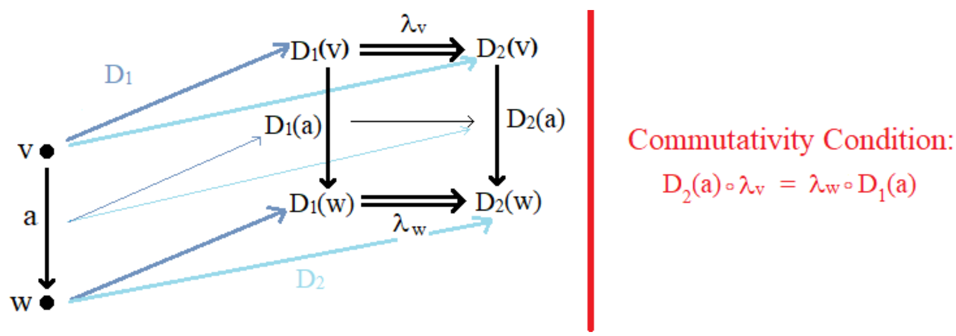
together with *path product* and the *identity arrows* introduced in the previous section in the trivial extension Σ_0.

• We may further impose a set of *commutativity conditions* $\boldsymbol{K}$ on our path categories. Whereby, we denote $\mathscr{P}(\boldsymbol{\Sigma/K})$ as above, except now the Hom collections become *equivalence classes* under the relation

$$\boldsymbol{f \sim g \leftrightarrow (f,g) \in K}.$$

Note: If $K = \emptyset$, we're back in the original definition.

• Def: Let $\boldsymbol{D_1 : \Sigma \to \mathcal{C}}$ and $\boldsymbol{D_2 : \Sigma \to \mathcal{C}}$ be two diagrams of the same type (works for $\boldsymbol{\Sigma/K}$ too). We define a **natural transformation of diagrams**, similar to that of functors, as a family of maps, $\boldsymbol{\{\lambda_v\}_{v \in Ve}}$, indexed over vertices making the following commute for each arrow:



• Def: We may then define the **Category of Diagrams of Type $\boldsymbol{\Sigma}$ in $\boldsymbol{\mathcal{C}}$**, denoted $[\boldsymbol{\Sigma}, \boldsymbol{\mathcal{C}}]$, to be the collection of all diagrams, $\boldsymbol{\{D : \Sigma \to \mathcal{C}\}}$, together with morphisms being *natural transformations* between diagrams, written for particular diagrams as say $\boldsymbol{Hom_{[\Sigma,\mathcal{C}]}(D,E) \equiv Nat(D,E)}$.

Analogously, one may define $[\boldsymbol{\Sigma/K}, \boldsymbol{\mathcal{C}}]$ and $[\mathscr{P}(\boldsymbol{\Sigma/K}), \boldsymbol{\mathcal{C}}]$, respectively the **Diagrams of Type $\boldsymbol{\Sigma/K}$ in $\boldsymbol{\mathcal{C}}$** and the **Functor Category between $\mathscr{P}(\boldsymbol{\Sigma/K})$ and $\boldsymbol{\mathcal{C}}$**.

[Exercise: What are the identity morphisms and how do we define composition in each of these three categories?]

Note: (p.38 [?]) Lastly, we mention the **Category of all *Diagram Schemes*** and diagrams between them (a.k.a. oriented graphs and morphisms thereof). We move on to discuss limits!

>> Pres of Functors | Equilizers <<

2. Limits of Diagrams

Observe the following technical definitions before the desired ones.

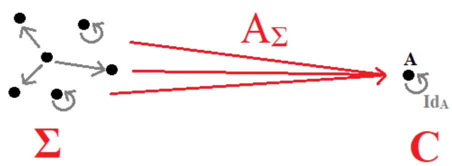
- -

• Def: (p.45 [?]) Let $\mathcal{C}$ be a category and $\boldsymbol{\Sigma}$ a diagram scheme.

For $A \in obj(\mathcal{C})$, let $A_\Sigma : \Sigma \to \mathcal{C}$ be the *constant diagram*:

$$\forall v \in Ve, \qquad A_\Sigma : v \mapsto A$$

$$\forall a \in Ar, \qquad A_\Sigma : a \mapsto Id_A$$

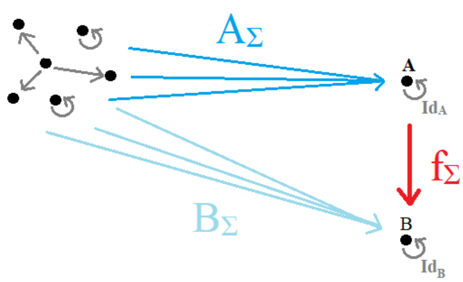


- -

• Def: Given a morphism $f : A \to B$ in $\mathcal{C}$, we get an *induced natural transformation* of constant diagrams:

$$f_\Sigma : A_\Sigma \to B_\Sigma$$

via declaring $f_\Sigma := \{(f_\Sigma)_v := f\}_{v \in Ve}$ such that $Id_B \circ f = f \circ Id_A$ (recall previous page).

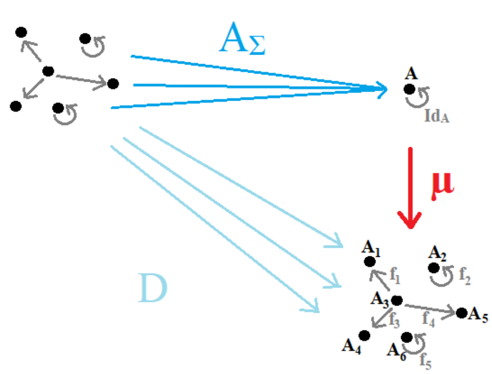


- -

• Def: If $D : \Sigma \to \mathcal{C}$ is a diagram, then a *nat. transformation from a constant diagram to D*, namely $\mu : A_\Sigma \to D$ consists of morphisms:

$$\{\mu_v :: A \to D(v)\}_{v \in Ve}$$

such that $\mu_{e(a)} \circ Id_A = D(a) \circ \mu_{o(a)}$ for all arrows in Ar.

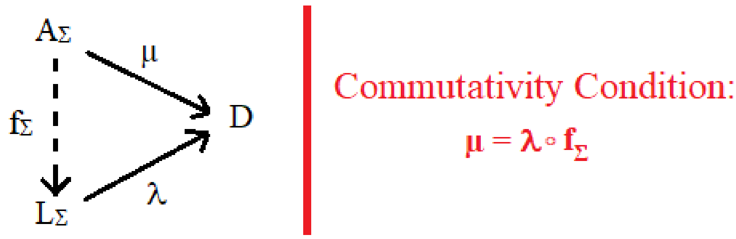


★ LIMITS

• <u>Def:</u> Given a diagram $\boldsymbol{D : \Sigma \to \mathcal{C}}$, we define the **limit** $(\boldsymbol{L, \lambda})$ **of the diagram** to be an object, $\boldsymbol{L} \in \boldsymbol{obj(\mathcal{C})}$ together with a natural transformation of diagrams, $\boldsymbol{\lambda} \in \boldsymbol{Nat(L_\Sigma, D)}$, satisfying the following:

Universal Property:

$\forall \boldsymbol{\mu} \in \boldsymbol{Nat(A_\Sigma, D)},\ \exists! \boldsymbol{f} \in \boldsymbol{Hom_{\mathcal{C}}(A, L)}$ such that:



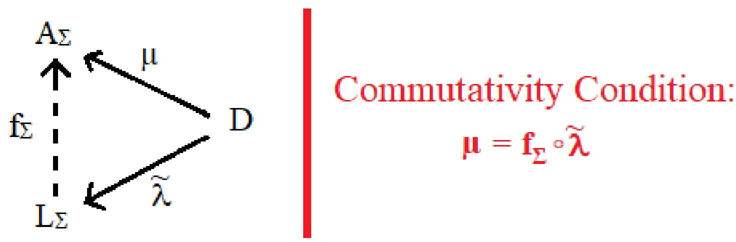
That is, every other natural transformation of diagrams from a constant diagram to D factors through $\boldsymbol{\lambda}$.

★ CO-LIMITS

• <u>Def:</u> (p.62-63 [?]) Dually, given a diagram $\boldsymbol{D : \Sigma \to \mathcal{C}}$, we define the **co-limit of the diagram** to be a pair $(\boldsymbol{\widetilde{L}, \widetilde{\lambda}})$, where $\boldsymbol{\widetilde{L}} \in \boldsymbol{obj(\mathcal{C})}$ and $\boldsymbol{\widetilde{\lambda}} \in \boldsymbol{Nat(D, L_\Sigma)}$, satisfying the following:

Universal Property:

$\forall \boldsymbol{\mu} \in \boldsymbol{Nat(D, A_\Sigma)},\ \exists! \boldsymbol{f} \in \boldsymbol{Hom_{\mathcal{C}}(L, A)}$ such that:



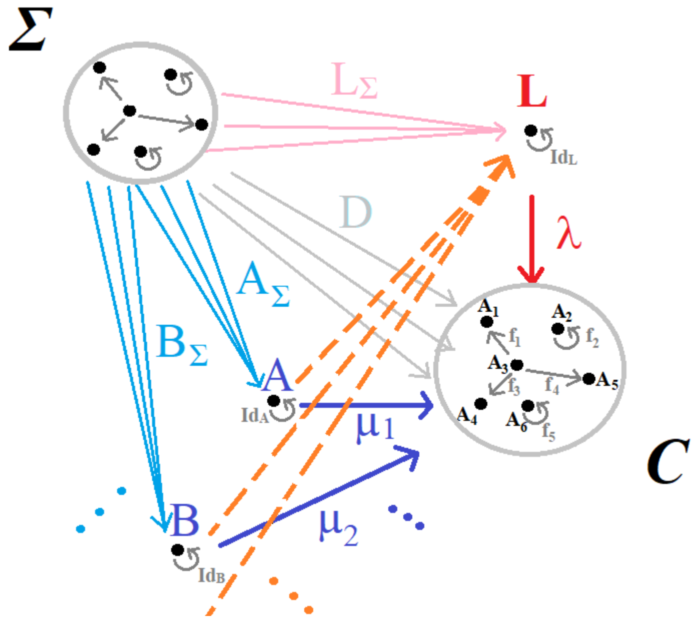
That is, every other natural transformation of diagrams from D into a constant diagram factors through $\boldsymbol{\widetilde{\lambda}}$.

Note: The tildes are not usually written it was to distinguish the limits from the colimits here notationally.

(Continues)

Notes:

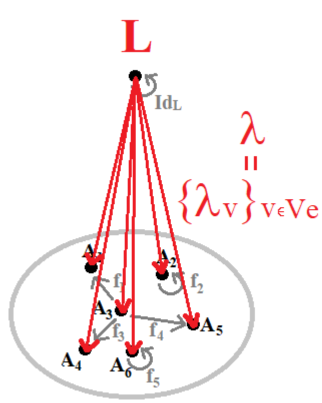
1.) Let's reinforce the distinction between a *limit of a diagram* and the limit's *universal property*:

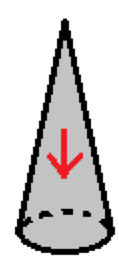


The limit is the pair (L, λ), all other such candidates of the form (A, μ_1), (B, μ_2), etc... have "arrows into (L, λ)". This feature gives rise to the alias: *terminal object*. Similarly, if we draw out the other picture [Exercise], we see the feature of co-limits is that there are arrows coming out of $(\widetilde{L}, \widetilde{\lambda})$ for all other candidates instead, giving colimits the alias: *initial object*. We will study initial and terminal objects in categories in the next section.

- -

2.) If we abstract and take a look at the pair (L, λ) a little closer, we get:





References [?] and [?] like to call this a **cone** (of type **Σ**). So that we have a good intuitive grasp on limits, we think of them as terminal objects in the **Category of Cones** (left to the interested reader to look up). Co-limits require the definition of **co-cones** (reverse arrow direction) etc.

(Continues)

Notes on Limits (Continued):

3.) Seemingly implicit in all of this is the diagram $\boldsymbol{D : \Sigma \to \mathcal{C}}$ for which $\boldsymbol{(L, \lambda)}$ is the limit. If we change $\boldsymbol{D}$ or $\boldsymbol{\Sigma}$ for that matter, we can look for different ***types of limits*** and limiting objects in categories.

Examples of Limits (See [**?**]): (We'll study some of these examples formally in the sequel.)

> Products
>> Powers
> Equilizers
>> Kernels
> Pullbacks
> Inverse Limits

Examples of Colimits:
> Initial Objects
> Coproducts
>> Copowers
> Coequilizers
>> Cokernels
> Pushouts
> Direct Limits

- -

4.) By the association we've seen between diagrams and functors, one can talk about **limits of functors**, $\boldsymbol{F : \mathcal{J} \to \mathcal{C}}$, indexed by some category $\boldsymbol{\mathcal{J}}$ - with appropriate adjustments in the definitions.
[Exercise: Rewrite the above constructions in terms of functors and fact check with [**?**].]

- -

5.) On Existence and Uniqueness of Limits and Colimits:
In the article [**?**], there is an *existence theorem for limits and colimits* stating criteria for a category to have them. Also, the author of the article states: "if a diagram does have a limit, then this limit is essentially unique: it is unique up to isomorphism".

• Def: (p.50 [**?**]) A category is called **complete** if every diagram has a limit. It is called **co-complete** if every diagram has a colimit.

- -

6.) Some advanced topics include:
i.) On Preservation of Limits by Functors,
ii.) Lifting of Limits, and
iii.) Creation and Reflection of Limits.

3. Object-Level Constructions

In this section, we rattle off some *special objects and morphisms* seen in the theory and gather them by local relevance to eachother.

Subsection Contents:

>> Jump to Section II.4

1: Invertible Morphisms, Isomorphisms, Retractions, Coretractions

• Def: (p.3 [?]) Suppose some category $\mathcal{C}$ is given and let $f \in Hom(A, B)$ and $g \in Hom(B, A)$. If it is the case that we have:

$$g \circ f = Id_A$$

then we say g **is a left inverse for** f and f **is a right inverse for** g.

If it is also the case that:

$$f \circ g = Id_B$$

then g and f become **two-sided inverses** or just **inverses**. Furthermore, considering f as fixed, existence of a two-sided inverse qualifies f with the alias of **isomorphism**.

[Exercise: Prove uniqueness of inverses when they exist.]

- -

• Def: (p.34 [?]) Alternative terminology: Given $r \in Hom(A, B)$ and $s \in Hom(B, A)$.
If it is the case that:

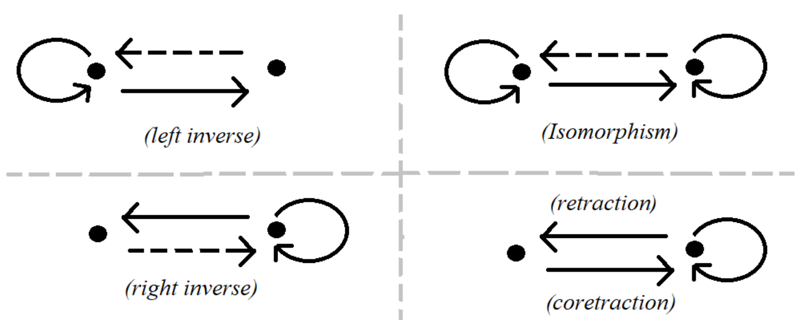
$$r \circ s = Id_B$$

then r **is called a retraction of** s and s **is called a co-retraction or section of** r.

Note: Fixing an $f \in Hom(A, B)$, sections of f are not unique in general. Likewise for retractions of f. A go to example exists in Differential Geometry, where we have the perspective of vector bundles with projections onto the manifold being the fixed maps and vector fields etc. being *sections* of the projection maps.

• Def: (p.3 [?]) Morphisms from an object to itself are called **endo-morphisms**.
That is, $f \in Hom(A, A) =: End(A)$. Elements of $End(A)$ that are also *isomorphisms* are called **auto-morphisms** and the subcollection of all such morphisms is denoted $Aut(A)$ for a given object A.

We may summarize the above conditions with diagrams schemes up to commutativity of paths.



2: Monics, Epics, and Bimorphisms

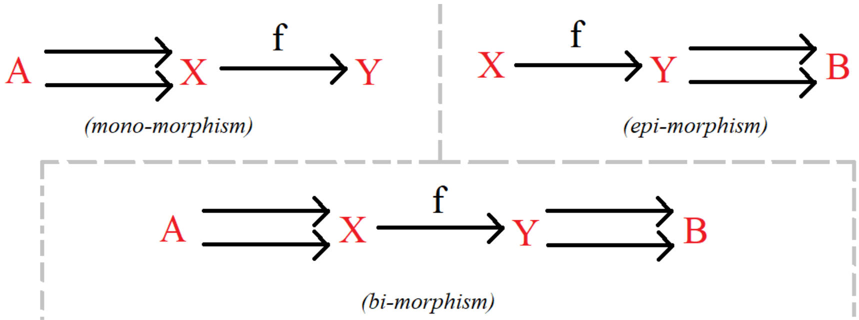


Fig: Schematics for memorization.

• Def: (p.32 [?]) We call $f \in Hom(X, Y)$ a **mono-morphism** (or **monic**) if it is **left-cancellative**.

That is, $\forall g_1, g_2 \in Hom(A, X)$:

$$(f \circ g_1 = f \circ g_2) \implies (g_1 = g_2).$$

- -

Equivalently, f is a monic if: $\forall A \in obj(\mathcal{C})$, the induced maps:

$$H^A(f) : Hom(A, X) \to Hom(A, Y)$$
$$g \mapsto f \circ g$$

are injective.

• Def: (p.33 [?]) Dually, $f \in Hom(X, Y)$ is an **epi-morphism** (or **epic**) if it is **right-cancellative**. That is, $\forall g_1, g_2 \in Hom(Y, B)$:

$$(g_1 \circ f = g_2 \circ f) \implies (g_1 = g_2).$$

- -

Equivalently, f is an epic if: $\forall B \in obj(\mathcal{C})$, the induced maps:

$$H_B(f) : Hom(Y, B) \to Hom(X, B)$$
$$g \mapsto g \circ f$$

are injective.

• <u>Def:</u> (p.34 [?]) A **bi-morphism** is a morphism that is **left and right cancellative** (i.e. it is both *monic* and *epic*).

<u>Notes:</u>
1.) In the category of *Sets*, monics are injective and epics are surjective. In general this is not the case. [Exercise: Explore this relationship. Hint: See [?] (p.32-34) or the Qual Problem here.]

2.) • <u>Def:</u> (p.34) Every *isomorphism* is a bi-morphism but not conversely. Categories for which the two notions are identical are called **balanced categories**.
[Exercise: Find a counter-example for the reverse direction. That is, find a bi-morphism that is not an iso-morphism.]

3: Subobjects and Quotient Objects

• Def: (p.4 [?]) If $\mathcal{C}$ is a category such that $\forall A, B$, we have either:

$$Hom(A,B) = \{f_{AB}\} \qquad \text{or} \qquad Hom(A,B) = \emptyset,$$

i.e. A and B are uniquely *comparable* or *not comparable* respectively, then the objects of $\mathcal{C}$ are given a *pre-order* relation, $\leq$, defined via:

$$A \leq B \;\leftrightarrow\; Hom(A,B) = \{f_{AB} : A \to B\}$$

since $\leq$ is clearly reflexive and transitive. Of course, if every two objects are comparable, we say $\leq$ is a *linear order* or *strong order*.

• Def: In the context of such a preordered $\mathfrak{U}$-class, we may define a new relation $\sim$ via:

$$A \sim B \;\leftrightarrow\; (A \leq B) \text{ and } (B \leq A)$$

which turns out to be an *equivalence relation* (reflexive, symmetric, and transitive). We then have **equivalence classes**:

$$[A]_\sim := \{B \mid B \sim A\},$$

which partitions the object collection of $\mathcal{C}$.

Def: (p.43-44 [?]) Let $X \in Obj(\mathcal{C})$ be given. Then define the morphism categories:

$$\mathcal{M}_X := \Big\{Obj(\mathcal{M}_X) := \{f \in Hom_{\mathcal{C}}(A,X) \mid f \text{ is monic, } A \in Obj(\mathcal{C})\}; \\ \{Hom_{\mathcal{M}_X}(f,g) := \{\varphi \in Hom_{\mathcal{C}}\Big(dom(f), dom(g)\Big) \mid f = g \circ \varphi\}\}_{f,g \in Obj(\mathcal{M}_X)}\Big\} \qquad \text{and}$$

$$\mathcal{E}^X := \Big\{Obj(\mathcal{E}^X) := \{f \in Hom_{\mathcal{C}}(X,B) \mid f \text{ is epic, } B \in Obj(\mathcal{C})\}; \\ \{Hom_{\mathcal{E}^X}(f,g) := \{\psi \in Hom_{\mathcal{C}}\Big(codom(f), codom(g)\Big) \mid \psi \circ f = g\}\}_{f,g \in Obj(\mathcal{E}^X)}\Big\},$$

where composition is inherited from $\mathcal{C}$. These are referred to respectively as the mono-morphisms with codomain X and the epi-morphisms with domain X.

[Exercise: Show that for any pair $f, g \in \mathcal{M}_X$ or $\mathcal{E}^X$, that $Hom(f,g)$ has at most one element (so that each yields a pre-ordered class and hence the objects of $\mathcal{M}_X$ and $\mathcal{E}^X$ get partitioned as well).]

★ Def: For a given $X \in obj(\mathcal{C})$ we define **subobjects of X** to be the elements of $Obj(\mathcal{M}_X)/\sim$ as defined above. That is, a subobject of X is just an equivalence class $[f : A \to X]_\sim$ of monomorphisms with codomain X.

Similarly, **quotient-objects of X**, denoted $[f : X \to B]_\sim \in Obj(\mathcal{E}^X)/\sim$, are just equivalence classes of epimorphisms with domain X under the relation described above.

Note: In concrete instances, one usually thinks of a subobject as the domain of a particular representative (i.e. the A above). Similarly for quotients... B above. But we stress here the extra data.

4: Initial, Terminal, and Zero Objects

• Def: (p.35 [?]) An **initial object**, $\boldsymbol{I} \in \boldsymbol{obj}(\boldsymbol{\mathcal{C}})$, has the property that:

$$\forall \boldsymbol{Y} \in \boldsymbol{obj}(\boldsymbol{\mathcal{C}}),\ \exists! \boldsymbol{f_Y} \in \boldsymbol{Hom}(\boldsymbol{I}, \boldsymbol{Y}).$$

Moreover, we require that $\forall \boldsymbol{Y},\ |\boldsymbol{Hom}(\boldsymbol{I}, \boldsymbol{Y})| = \mathbf{1}$.

- -

• Def: A **terminal object**, $\boldsymbol{T} \in \boldsymbol{obj}(\boldsymbol{\mathcal{C}})$, has the property that:

$$\forall \boldsymbol{X} \in \boldsymbol{obj}(\boldsymbol{\mathcal{C}}),\ \exists! \boldsymbol{f_X} \in \boldsymbol{Hom}(\boldsymbol{X}, \boldsymbol{T}).$$

Moreover, we require that $\forall \boldsymbol{X},\ |\boldsymbol{Hom}(\boldsymbol{X}, \boldsymbol{T})| = \mathbf{1}$.

- -

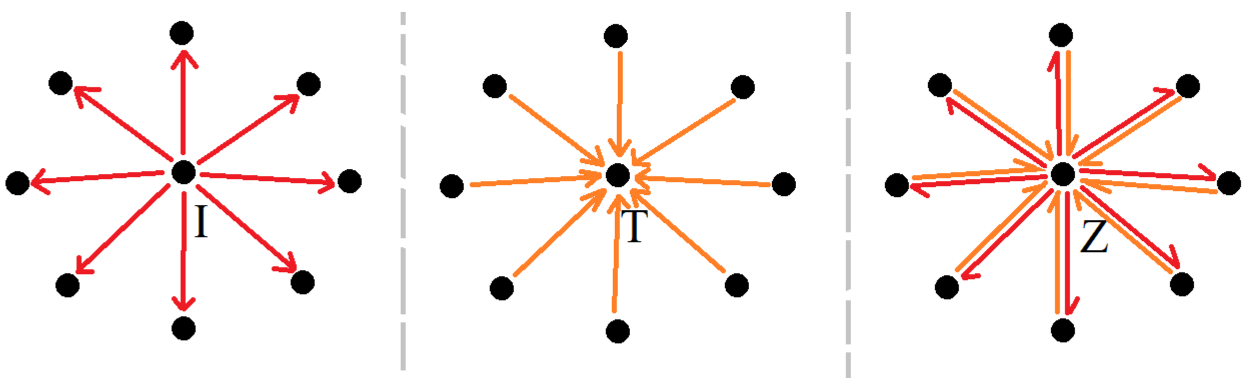
• Def: A **zero object**, $\boldsymbol{Z} \in \boldsymbol{obj}(\boldsymbol{\mathcal{C}})$, has the property that:

$$\forall \boldsymbol{W} \in \boldsymbol{obj}(\boldsymbol{\mathcal{C}}),\ \exists! \boldsymbol{f_W} \in \boldsymbol{Hom}(\boldsymbol{W}, \boldsymbol{Z}) \text{ and } \exists! \boldsymbol{g_W} \in \boldsymbol{Hom}(\boldsymbol{Z}, \boldsymbol{W}).$$

And require $\forall \boldsymbol{W},\ |\boldsymbol{Hom}(\boldsymbol{W}, \boldsymbol{Z})| = \mathbf{1} = |\boldsymbol{Hom}(\boldsymbol{Z}, \boldsymbol{W})|$.

That is, zero objects are both initial and terminal.

Notes: Rephrased, initial objects have arrows eminating from them to every other object in the category, terminal objects have arrows converging to them for every other object in the category. Think sources and sinks respectively. The zero object is like a trivial structure that "includes" in every other object and can be "collapsed to" from every other object. This is of course a notion specific to our intuition in say the category of groups.



(initial) (terminal) (zero)

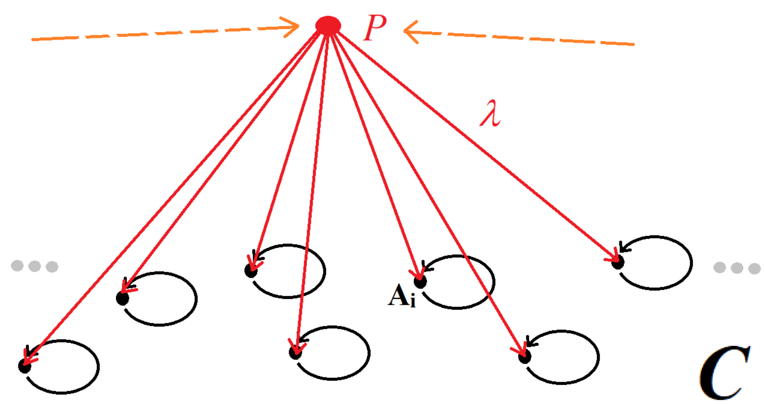
Furthermore, as the author notes on (p.35), "there is a unique isomorphism between each pair of zero objects of a category. Hence if zero objects exist in a category, we fix one and denote it by **0**."

5: Product and Coproduct Objects

• Def: (p.49 [?]) A **product** in a category $\mathcal{C}$ is a *limit of discrete type.*

- -

That is, a product is a *terminal cone* whose base has a *pattern* given by (only) objects and identity morphisms.



Recall: More explicitly, $(P, \lambda) \in Obj(\mathcal{C}) \times Nat\Big(P_\Sigma : \Sigma \mapsto P, D : \Sigma \to \mathcal{C}\Big)$ and ∀ other such pairs (Q, μ), $\exists! f \in Hom_{\mathcal{C}}(Q, P)$ such that $\mu = \lambda \circ f_\Sigma$.

We refer to the object P as the **product object**.

- -

• Def: Dualizing the above, we get that **coproducts** are just *colimits of discrete type.*
That is, *initial co-cones* with discrete base. Hence defining **coproduct objects** as the $\widetilde{P}$ in $(\widetilde{P}, \widetilde{\lambda})$.

(Continues)

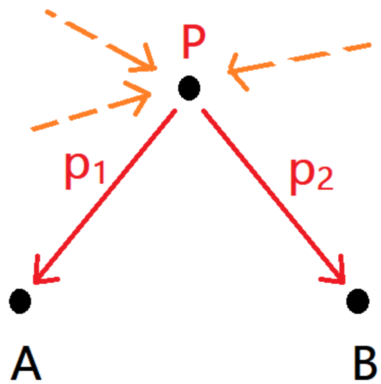
<u>Note:</u> We usually denote products/coproducts in binary notion or prefix notation with indexing. That is for example:

$$\boldsymbol{P = A \prod B} \qquad \text{or} \qquad \prod_{i \in I} A_i$$

$$\boldsymbol{\widetilde{P} = A \coprod B} \qquad \text{or} \qquad \coprod_{i \in I} A_i$$

<u>Examples:</u>

1.) Listing out the limit $\boldsymbol{(P, \lambda) := (P, \{p_1, p_2\})}$ in a special case where the natural transformation $\boldsymbol{\lambda = \{\lambda_X\}_{X \in obj(\mathcal{C})}}$ consists of two ***projection-esque* maps**, we get:



(id_X's omitted)

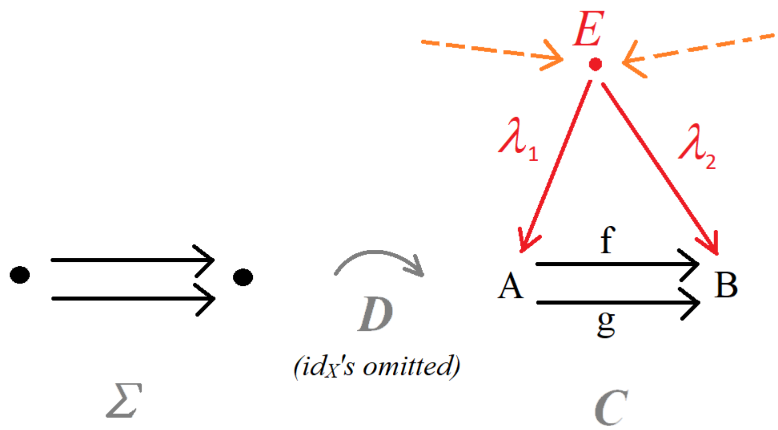
which can of course be upgraded to larger cardinalities for the base of the cone [Exercise].

2.) Coproducts have a similar description.

3.) Now we can talk about *product categories* $\boldsymbol{\mathcal{C} \times \mathcal{D}}$ and *multi-functors* $\boldsymbol{F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}}$ etc. in $\boldsymbol{cat, Cat, CAT}$ [See: Section II.4].

6: Equilizers and Coequilizers

• Def: (p.47-48 [?]) Consider the limit (E, λ) of the diagram $D : \Sigma \to \mathcal{C}$ displayed below:



More precisely,
$E \in obj(\mathcal{C})$ and $\lambda \in Nat(E_\Sigma, D)$ are such that:

Universal Property:
$\forall (F, \mu) \in Obj(\mathcal{C}) \times Nat(F_\Sigma, D)$, $\exists!\ h : F \to E$ for which $\mu = \lambda \circ h_\Sigma$.

- -

Note:
The *commutativity conditions* of the natural transformation λ imply that:

$$f \circ \lambda_1 = \lambda_2 \qquad \text{and} \qquad g \circ \lambda_1 = \lambda_2$$

[Review Section II.1.2 for clarification]. Since f and g are given, this says that this limit is specified by a single map $\lambda_1 : E \to A$ together with the equality: $f \circ \lambda_1 = g \circ \lambda_1$ and an appropriate version of the universal property (other such μ_1's factor through λ_1).

Under these conditions, the map $\lambda_1 : E \to A$ is called an **equilizer of the pair** (f, g).

• Def: (p.64 [?]) Dualizing the above (i.e. flipping arrows in the cone not in the base),
we get that a **co-equilizer of** $f, g : A \to B$ is specified by:

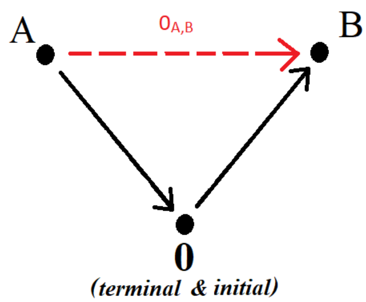
$$\widetilde{\lambda}_2 : B \to \widetilde{E} \qquad \text{such that} \qquad \widetilde{\lambda}_2 \circ f = \widetilde{\lambda}_2 \circ g$$

and all other such $(\widetilde{\mu}_2 : B \to \widetilde{F})$'s factor uniquely through $\widetilde{\lambda}_2$ as in $\widetilde{\mu}_2 = \widetilde{h}_\Sigma \circ \widetilde{\lambda}_2$.

(Next Page)

Kernels and Co-kernels

• Def: (p.36 [?]) Recall if a category has zero objects it has a unique zero object, $\mathbf{0}$, specified up to isomorphism. By the initial and terminal object properties, we have for any pair of objects $A, B \in obj(\mathcal{C})$, there is a unique map, $0_{A,B} : A \to B$, called the **zero morphism from A to B**, such that the following commutes:



★ Def: (p.48 [?]) In a category with zero object(s), the special case of an equilizer of the pair $f : A \to B$ and $\mathbf{0} : A \to B$ is called the **kernel of f**. With the above considerations, the kernel may be denoted by $(K, ker(f))$ in limit notation or by

$$ker(f) : K \to A \text{ such that } f \circ ker(f) = 0_{KB}$$

and any other such $g : G \to A$ with $f \circ g = 0_{GB}$ is such that $\exists!\ h : G \to K$ with $g = ker(f) \circ h$.

★ Def: (p.64) Similarly we define **co-kernels**, $(\widetilde{K}, coker(f))$, of morphisms $f : A \to B$ by:

$$coker(f) : B \to \widetilde{K} \text{ such that } coker(f) \circ f = 0_{A\widetilde{K}}$$

subject to the *initial* universal property that any other such $g : B \to G$ with $g \circ f = 0_{AG}$ is such that $\exists!\ h : \widetilde{K} \to G$ with $g = h \circ coker(f)$.

Note: The last two universal properties and definitions of kernel and cokernel get used extensively in Homological Algebra.

7: Projective and Injective Objects

Note the symbology used for monics and epics below. The unusual arrows indicate the double morphism properties at the beginning or the end (as we have seen). Writing out explicitly aids in visualizing duality.

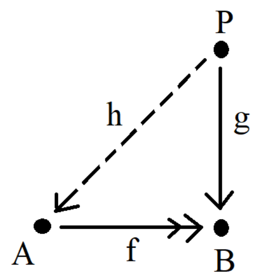
• Def: (p.89 [?]) $P \in obj(\mathcal{C})$ is called **projective** if the corresponding covariant Hom functor $H^P(\bullet)$ *takes epi-morphisms to epi-morphisms.*

- -

More explicitly, if $f : A \to B$ is an epimorphism, then so is $H^P(f) : Hom(P, A) \to Hom(P, B)$.

But since the latter is just a morphism in the category of *Sets* for which epics are surjective maps, this implies that we call P a projective object if for any given epic $f : A \to B$, we have:

$$\forall g \in Hom(P, B), \exists h \in Hom(P, A), \text{ such that } f \circ h = g$$



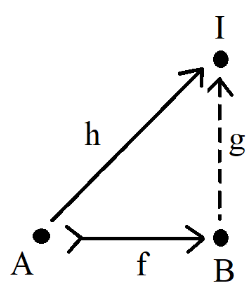
• Def: (p.90) Dually, we call $I \in obj(\mathcal{C})$ an **injective object** if the corresponding contravariant Hom functor $H_I(\bullet)$ *takes mono-morphisms to epi-morphisms.*

- -

Explicitly, if $f : A \to B$ is any monomorphism, then $H_I(f) : Hom(B, I) \to Hom(A, I)$ is surjective.

I.e. for a given monic $f : A \to B$, we have:

$$\forall h \in Hom(A, I), \exists g \in Hom(B, I), \text{ such that } g \circ f = h$$



8: Generators and Cogenerators

The following are bare definitions for the sake of seeing them. They crop up in more advanced theory building up to a "Representation Theorem" (pp.91-95) [?]). See also Ch.5 of [?]

• Def: (p.91 [?]) Let $\mathcal{C}$ be a category and $\mathfrak{G}$ be a *set* of objects of $\mathcal{C}$. If:

$$\forall A, B \in obj(\mathcal{C}) \text{ and } \forall f, g \in Hom(A, B),$$

$$\exists h \in Hom(G, A) \text{ for some } G \in \mathfrak{G} \text{ with } f \circ h \neq g \circ h,$$

then we say $\mathfrak{G}$ is a **generating set**. An object G is called a **generator** if $\{G\}$ is a generating set.

• Def: (p.92 [?]) Let $\mathcal{C}$ be a category and $\widetilde{\mathfrak{G}}$ be a *set* of objects of $\mathcal{C}$. $\widetilde{\mathfrak{G}}$ is called a **cogenerating set** if it is a generating set for $\mathcal{C}^{op}$. That is, if the following is satisfied:

$$\forall A, B \in obj(\mathcal{C}) \text{ and } \forall f, g \in Hom(A, B),$$

$$\exists \widetilde{h} \in Hom(B, \widetilde{G}) \text{ for some } \widetilde{G} \in \widetilde{\mathfrak{G}} \text{ s.t. } \widetilde{h} \circ f \neq \widetilde{h} \circ g.$$

The object $\widetilde{G}$ is called a **cogenerator** if $\{\widetilde{G}\}$ is a cogenerating set.

And that concludes our discussion on special objects and morphisms seen at the individual category level. Next up, we consider the settings of cat, Cat, CAT!

4. Category-Level Constructions

In this section we explore some universal algebraic like constructions.

>> Jump to Section III.1

1: Subcategories

• Def: (p.4 [?]) Given a category $\mathcal{C}$, let us define a new category , $\mathcal{D}$, by declaring:

(i) $Obj(\mathcal{D}) \subseteq Obj(\mathcal{C})$,

(ii) $\forall A, B \in Obj(\mathcal{C}),\ Hom_{\mathcal{D}}(A,B) \subseteq Hom_{\mathcal{C}}(A,B)$,

(iii) $\forall A \in Obj(\mathcal{D}),\ Id_A \in Hom_{\mathcal{D}}(A,A)$, and

(iv) $\forall A, B, C \in Obj(\mathcal{D}),\ \forall f \in Hom_{\mathcal{D}}(A,B), \forall g \in Hom_{\mathcal{D}}(B,C),\ g \circ f \in Hom_{\mathcal{D}}(A,C)$.

That is, we take sub-collections from the objects and morphisms of $\mathcal{C}$, include all identities, and close under "restricted" composition (associativity is inherited). The new category obtained in this way is called a **subcategory** of $\mathcal{C}$ and the relationship can be indicated by $\mathcal{D} \subseteq \mathcal{C}$.
There is also an associated **inclusion functor $I : \mathcal{D} \to \mathcal{C}$**.

[Exercise (Problem): In cat, Cat, CAT, where the objects are categories and the morphisms are functors between them, we have the notion of *sub-objects* of a fixed category $\mathcal{C}_0$,

$$[F : \mathcal{D} \to \mathcal{C}_0]_{\sim},$$

as equivalence classes of monomorphic functors with a common codomain (see Section II.3.3). How does this definition reconcile with the one provided above for *subcategories* in this context?]

Here are some more definitions related to subcategories:

• Def: In the event that we have *equality* in item (ii) above for every pair of objects in $\mathcal{D}$, we call $\mathcal{D}$ a **full subcategory** of $\mathcal{C}$. This terminology aligns with our notion of *(full) functors* from Section I.2.1 in the sense that the inclusion functor from a full subcategory to its super-category is full.

• Def: (p.25 [?]) Recall that a functor $F : \mathcal{D} \to \mathcal{C}$ is an *embedding* if $F_2 : Hom(\mathcal{D}) \to Hom(\mathcal{C})$ is injective. In such a case, we call $F(D)$ an **embedded subcategory** of $\mathcal{C}$.

Note: It is not enough for the functor to be *faithful* to make $F(\mathcal{D})$ into a category as Schubert shows on (p.25) with a counterexample.

• Def: (p.170 [?])
A category $\mathcal{K}$ is called **reduced** if any two isomorphic objects are identical. A subcategory $\mathcal{K} \subseteq \mathcal{C}$ is called a **skeleton of $\mathcal{C}$** if $\mathcal{K}$ is *reduced* and if the inclusion functor $\iota : \mathcal{K} \to \mathcal{C}$ is an *equivalence*. That is,

$$\exists G : \mathcal{C} \to \mathcal{K}, \text{ for which } G \circ \iota \cong Id_{\mathcal{K}} \text{ and } \iota \circ G \cong Id_{\mathcal{C}}$$

2: Quotients of Categories

Recall: The objects of a category can be considered as a subcollection of the morphisms by way of the identities. In this regard, any relation on the objects can be manifest as a relation on the morphisms (just trivially extend the relation by inclusion). So WLOG below...

We want to construct quotients of categories by imposing certain "congruence relations" on them.

• Def/Prop: (p.42 [?]) Suppose we have a binary relation:

$$\sim \;\; \subseteq \;\; \boldsymbol{Mor}(\mathcal{C}) \times \boldsymbol{Mor}(\mathcal{C})$$

having the following properties:

1.) $$\sim \;\; := \bigcup_{A,B \in Obj(\mathcal{C})} \sim_{AB},$$

where each $\sim_{AB} \;\; \subseteq \;\; \boldsymbol{Hom}(A,B) \times \boldsymbol{Hom}(A,B)$ is an *equivalence relation* (i.e. they are *reflexive*, *symmetric*, and *transitive*) and

2.) $\forall\, f, f' \in \boldsymbol{Hom}(A,B),$ and $\forall\, g, g' \in \boldsymbol{Hom}(B,C)$, if the compositions are defined:

$$\text{“ } f \sim f' \text{ and } g \sim g' \implies g \circ f \sim g' \circ f' \text{ ”}.$$

- -

We may list the **quotient category** as the two collections:

$$\mathcal{C}/\!\sim \;:=\; \Big\{ \boldsymbol{Obj}(\mathcal{C}/\!\sim) := \boldsymbol{Obj}(\mathcal{C}),$$
$$\boldsymbol{Mor}(\mathcal{C}/\!\sim) := \left\{ \boldsymbol{Hom}(A,B)/\!\sim_{AB} \;\middle|\; A, B \in \boldsymbol{Obj}(\mathcal{C}) \right\} \Big\}$$

together with a well-defined composition operation and identities, respectively:

$$[g]_\sim \,\tilde{\circ}\, [f]_\sim := [g \circ f]_\sim \qquad \text{and} \qquad [Id_A]_\sim, \text{ for each object } A.$$

Moreover $P : \mathcal{C} \to \mathcal{C}/\!\sim;\;\; f \mapsto [f]_\sim$ defines a **projection functor**.

[Exercise (Problem): In $\boldsymbol{cat}, \boldsymbol{Cat}, \boldsymbol{CAT}$, where the objects are categories and the morphisms are functors between them, we have the notion of *quotient-objects* of a fixed category $\mathcal{C}_0$,

$$[F : \mathcal{C}_0 \to \mathcal{D}]_{\dot\sim},$$

as equivalence classes of epi-morphic functors with a common domain (see Section II.3.3). How does this definition reconcile with the one provided above for *quotients of categories* in this context?]

3: Product and Coproduct Categories

• Def/Prop: (p.10 [?]) Given two categories $\mathcal{C}$ and $\mathcal{D}$, the (binary) **product category**, is given by applying the set/class-theoretic (Cartesian) product to the object and morphism collections:

$$\mathcal{C}\dot{\times}\mathcal{D} := \left\{ Obj(\mathcal{C}\dot{\times}\mathcal{D}) := Obj(\mathcal{C}) \times Obj(\mathcal{D}),\ Hom(\mathcal{C}\dot{\times}\mathcal{D}) := Hom(\mathcal{C}) \times Hom(\mathcal{D}) \right\}$$

where for morphisms $(f, \alpha) : (A, \mathscr{A}) \to (B, \mathscr{B})$ and $(g, \beta) : (B, \mathscr{B}) \to (C, \mathscr{C})$, composition is defined component-wise by:

$$(g, \beta)\tilde{\circ}(f, \alpha) := (g \circ f,\ \beta \circ \alpha) : (A, \mathscr{A}) \to (C, \mathscr{C})$$

and the identities are given by:

$$Id_{(A,\mathscr{A})} := (Id_A, Id_{\mathscr{A}}).$$

Associativity is inherited from the component compositions' property.

Notes:
1.) The symbolism above is for display purposes here not really conventional. The dot on times and tilde on composition are for differentiating the new symbols defined in terms of the old. In context you will not see that.

2.) Also technically, we should list Π instead of $\times$, appealing to the following:

[Exercise: In **cat, Cat, CAT**, we have *discrete-type limits* also characterizing product objects (product categories). Show that our new definition above satisfies being the object of a discrete limit in cat, Cat, CAT.]

3.) We can upgrade (binary) to (arbitrary indexed) using that of the Cartesian operation.

• Def/Prop: Given two categories $\mathcal{C}$ and $\mathcal{D}$, we define the (binary) **coproduct category** using the set/class-theoretic (disjoint union) to the object and morphism collections:

$$\mathcal{C}\dot{\amalg}\mathcal{D} := \left\{ Obj(\mathcal{C}\dot{\amalg}\mathcal{D}) := Obj(\mathcal{C}) \amalg Obj(\mathcal{D}),\ Hom(\mathcal{C}\dot{\amalg}\mathcal{D}) := Hom(\mathcal{C}) \amalg Hom(\mathcal{D}) \right\}$$

where composition is defined piecewise as:

$$g\tilde{\circ}f := \begin{cases} g \circ_1 f, & \text{if } f, g \in Hom(\mathcal{C}) \text{ are compatible} \\ g \circ_2 f & \text{if } f, g \in Hom(\mathcal{D}) \text{ are compatible} \\ \text{Not defined otherwise} \end{cases}$$

and identities are just collected. Associativity inherited; (binary) generalize-able via disjoint union operation.

[Exercise: Show that our new definition above satisfies being the object of a discrete co-limit in cat, Cat, CAT.]

<< PART III >>

1. Qualifier Problems and Proofs

The following problems (except 1) come from Hawaii University's 2016-2018 Qualifying Exams.

>> Jump to Advanced Material

Problem 1.) **Prove Co-variance of the Covariant Hom Functor:**

$$H^A : \mathcal{C} \to Sets$$
$$H^A(Z) := Hom(A, Z)$$
$$H^A(f) :: Hom(A, X) \to Hom(A, Y); \ \varphi \mapsto f \circ \varphi$$

where of course Z and $f : X \to Y$ range over objects and morphisms in $\mathcal{C}$.

<u>Proof:</u> Recall a functor $F : \mathcal{C} \to \mathcal{D}$ is called *covariant* if

(i) $f \in Hom_{\mathcal{C}}(X, Y) \implies F(f) \in Hom_{\mathcal{D}}(F(X), F(Y))$ and

(ii) $\forall f \in Hom(X, Y), \forall g \in Hom(Y, Z)$, we have $F(g \circ f) = F(g) \circ F(f)$.

We proceed to prove both of these.

- -

(i) Let $f \in Hom(X, Y)$ be given. Then since $Hom(A, X) =: H^A(X)$ and $H(A, Y) =: H^A(Y)$, we have by definition that $H^A(f) : H^A(X) \to H^A(Y) \left(\in Hom_{Sets}(H^A(X), H^A(Y)) \right)$. □

(ii) Since the image of the functor is in the category Set, we know how to show *equality of morphisms*. Particularly, we show that two morphisms have equal images on their domains.

Let $f : X \to Y$, $g : Y \to Z$, then:

$$H^A(g \circ f) : Hom(A, X) \to Hom(A, Z)$$
$$\varphi \mapsto (g \circ f) \circ \varphi$$

$$H^A(g) : Hom(A, Y) \to Hom(A, Z)$$
$$\psi \mapsto g \circ \psi \qquad \text{and}$$
$$H^A(f) : Hom(A, X) \to Hom(A, Y).$$
$$\varphi \mapsto f \circ \varphi$$

From the last two items, we get that for $\varphi : A \to X$, $[H^A(g) \circ H^A(f)](\varphi) = g \circ (f \circ \varphi)$. By the *associativity axiom* of categories:

$$(g \circ f) \circ \varphi = g \circ (f \circ \varphi)$$

and hence we conclude that $[H^A(g \circ f)](\varphi) = [H^A(g) \circ H^A(f)](\varphi)$.

By arbitrariness of $\varphi \in Hom(A, X)$, we get the desired equality of morphisms. ■

>> Back to Section <<

Problem 2.)
(a) Let $\mathcal{C}$ be a category and let C be a fixed object of $\mathcal{C}$. For an object A of $\mathcal{C}$, let $F_C(A) = Hom_{\mathcal{C}}(A, C)$ be the set of morphisms in $\mathcal{C}$ from A to C. Show that F_C gives a *contravariant* functor from $\mathcal{C}$ to the category of $Sets$. In particular, for a morphism $f : A \to B$ of $\mathcal{C}$, say what $F_C(f)$ is.

(b) Suppose $\mathcal{C}$ is the category of $Sets$ and the $C = \{0, 1\}$. Show that for every set A, there is a bijection between $F_C(A)$ and the set of subsets of A.

Proof:
a.) In a long winded way, $F_C(f)$ describes the contravariant Hom functor we have seen: $H_A(\cdot)$ (dual to in problem 1), except with A switched with C.

To show F_C defines a *functor*, we need to show it is a *bi-map* and that *identities are preserved*.

If we take $Hom(A, C)$ and $Hom(B, C)$ and try to create a morphism out of them using $f : A \to B$, we see the only way is to pre-compose with f. That is, if $\varphi \in Hom(B, C)$ then $\varphi \circ f : A \to C$. So we take as definition:

$$F_C(f) : Hom(B, C) \to Hom(A, C)$$

$$F_C(f) : \varphi \mapsto \varphi \circ f$$

for the second part of the bi-map; The first part was given to us ($F_C(A) := Hom_{\mathcal{C}}(A, C)$).

Taking $f := Id_A$, we see then that $F(Id_A) : \varphi \mapsto \varphi \circ Id_A = \varphi$. So $F_C(Id_A) = Id_{F_C(A)}$.

- -

Proving *contravariance* amounts to showing:

$$F_C(f) \in Hom_{Sets}(F_C(Y), F_C(X)) \qquad \text{and} \qquad F_C(g \circ f) = F_C(f) \circ F_C(g),$$

where $f : X \to Y$ and $g : Y \to Z$ and the equality is for morphisms in $Sets$.

The first statement is clear since the variables were just switched in the definition.
Now, the second statement falls out as before noting that:

$$[F_C(g \circ f)](\varphi) = \varphi \circ (g \circ f) \qquad \text{and} \qquad [F_C(f) \circ F_C(g)](\varphi) = (\varphi \circ g) \circ f.$$

and quoting associativity and arbitrariness of φ. ■

(Continues)

b.) Decoding, WTS

$$\forall A \left(\exists f_A : Hom_{Sets}(A, \{0,1\}) \xrightarrow{\cong} \mathscr{P}(A) \right)$$

where $\mathscr{P}(A)$ is just the power set of A.

- -

Suppose A and $\varphi : A \to \{0,1\}$ are given. Then define:

$$f_A(\varphi) := \varphi^{-1}(1)$$

that is, the pullback of $\mathbf{1}$ (the subset of elements in A that map to $\mathbf{1}$ through φ). So varying φ gives us different subsets in the power set.

We have existence, now WTS that each f_A is *bijective*.

• Injectivity:
Suppose $\varphi, \psi \in Hom(A, \{0,1\})$ are such that $f_A(\varphi) = f_A(\psi)$. Then we have $\varphi^{-1}(1) = \psi^{-1}(1)$, but this then determines $\varphi^{-1}(0) = \psi^{-1}(0)$ since we just take complements. So we have $\varphi \equiv \psi$.

• Surjectivity:
Lastly, suppose a subset $S \subseteq A$ is given. Then defining the function $\delta_S := \begin{cases} 1 & \text{on } S \\ 0 & \text{otherwise} \end{cases}$ shows $\exists \varphi \in Hom(A, \{0,1\})$ such that $f_A(\varphi) = S$ and we're done. ■

Problem 3.) **For a group G, let $\mathcal{C}_G$ be the associated groupoid, i.e. the category $\mathcal{C}_G$ that has exactly one element, denoted $\bullet_G$, whose morphisms are described as follows: for each $g \in G$ there is an *isomorphism* $f_g : \bullet_G \to \bullet_G$ and the composition is defined by $f_g \circ f_h = f_{gh}$ using the group operation (i.e. $Hom_{\mathcal{C}_G}(\bullet_G, \bullet_G)$ is isomorphic to G as a group).**

(a) **Suppose G and H are groups. Show that giving a functor $F : \mathcal{C}_G \to \mathcal{C}_H$ is the same as giving a group homomorphism $\varphi : G \to H$.**

(b) **Given two group homomorphisms $\varphi, \psi : G \to H$, say that φ and ψ are *conjugate* if there is an $h \in H$ such that for all $g \in G$:**

$$h\varphi(g)h^{-1} = \psi(g).$$

From the previous part, given $\varphi : G \to H$, there is corresponding $F_\varphi : \mathcal{C}_G \to \mathcal{C}_H$. Show that φ and ψ are conjugate iff there is a natural transformation $\eta : F_\varphi \to \mathcal{G}_\psi$.

Proof: First let's summarize. We defined a **groupoid** (the categorization of a group) to be:

$$\mathcal{C}_G := \left\{ Obj(\mathcal{C}_G) = \{\bullet_G\}, \;\; Mor(\mathcal{C}_G) = \{iso's\ f_g : \bullet_G \to \bullet_G\} \cong G \right\}$$

together with composition of morphisms defined as $f_g \circ f_h = f_{gh}$. We have then $g \mapsto f_g$ is a group iso.

- -

a.) Suppose we are given a (covariant) functor $F : \mathcal{C}_G \to \mathcal{C}_H$. This is a bi-map:

$$Obj(\mathcal{C}_G) = \{\bullet_G\} \to \{\bullet_H\} = Obj(\mathcal{C}_H)$$

$$Mor(\mathcal{C}_G) = \{iso's\ f_g : \bullet_G \to \bullet_G\} \to \{iso's\ f_h : \bullet_H \to \bullet_H\} = Mor(\mathcal{C}_H)$$

such that $F(f_g \circ f_h) = F(f_g) \circ F(f_h)$ and $F(f_{1_G}) = f_{1_H}$.

But since the morphism collections are both isomorphic to G and H respectively and there is only one possible map on the object sets, this says that F specifies a map between G and H which preserves the group operations and identities, i.e. specifies a group homomorphism. ■

(Continues)

b.) Let $\varphi, \psi : G \to H$ be two *conjugate* group homomorphisms. Then:

$$\exists h \in H, \forall g \in G,\ h\varphi(g)h^{-1} = \psi(g) \qquad (\bigstar)$$

WTS this doubly implies existence of a *natural transformation* between the corresponding functors:

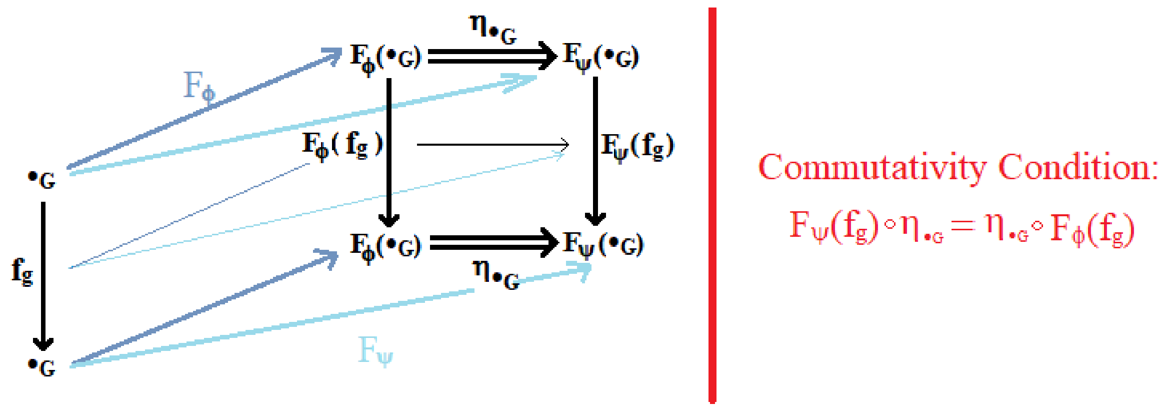
$$F_\varphi, F_\psi : \mathcal{C}_G \to \mathcal{C}_H.$$

Since there is only one object in $\mathcal{C}_G$, we may specify a natural transformation of the functors with one morphism. That is, we may define:

$$\eta : F_\varphi \to F_\psi$$

$$\eta = \{\eta_{\bullet_G}\}$$

such that the following diagram commutes for arbitrary f_g:



Since $\eta_{\bullet_G}$ is a morphism in $\mathcal{C}_H$, we know it is an isomorphism and hence has a two-sided inverse. So we can rewrite the commutativity conditions as:

$$\forall f_g, \qquad \eta_{\bullet_G} \circ F_\varphi(f_g) \circ \eta_{\bullet_G}^{-1} = F_\psi(f_g)$$

Taking $(\bigstar)$ and passing to the groupoid statement via the isomorphism $h \to f_h$ etc. provides the correspondence. ■

Problem 4.) **Let Set_* be the category of pointed sets, i.e. the category whose objects are pairs (X, x_0) where X is a set and x_0 is an element of X called the *basepoint*, and whose morphisms $(X, x_0) \to (Y, y_0)$ are functions $f : X \to Y$ such that $f(x_0) = y_0$.**

(a) Given two pointed sets (X, x_0) and (Y, y_0), show that $(X \times Y, (x_0, y_0))$ gives their product in the category of pointed sets.

(b) Given two pointed sets (X, x_0) and (Y, y_0), define their *wedge sum* $X \vee Y$ to be their disjoint union with the basepoints identified, i.e. the pair $((X \amalg Y)/\sim, [x_0])$, where $X \amalg Y$ is the usual disjoint union, $\sim$ is the equivalence relation where $x_0 \sim y_0$ and all other points are only equivalent to themselves, and $(X \amalg Y)/\sim$ denotes the set of equivalence classes and $[x_0]$ the equivalence class of x_0. Show that the wedge sum is the coproduct in the category of pointed sets.

Proof:

We have: $Set_* := \Big\{ Obj(Set_*) := \{(X, x_0) \mid X \text{ is a set and } x_0 \in X\},$

$$Mor(Set_*) := \{f : X \to Y \mid f(x_0) = y_0\} \Big\}$$

- -

a.) WTS the product in Set_* is the discrete limit $(L, \lambda) := \Big((X \times Y, (x_0, y_0)),\ p_1,\ p_2\Big)$, where

$$p_1 : \big(X \times Y, (x_0, y_0)\big) \to (X, x_0)$$

and

$$p_2 : \big(X \times Y, (x_0, y_0)\big) \to (Y, y_0)$$

are the usual projection maps.

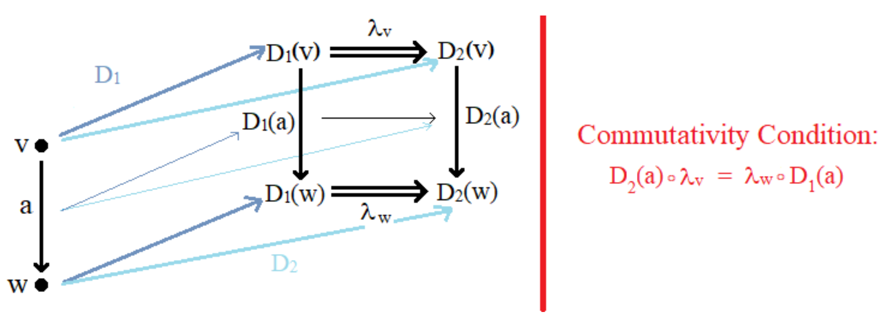
Well, aside from showing λ gives a *natural transformation of diagrams*, i.e. showing

$$\lambda = \{p_1, p_2\} \in Nat_{[\Sigma, Set_*]}(L_\Sigma, D),$$

the other part of the problem is to show (L, λ) satisfies the *terminal object universal property.*

- -

(i.) Recall the set up for natural transformations of diagrams:



(Continues)

4a.) i.) (Continued):
If we note the *type* of the diagram is $(\boldsymbol{Ve} : \mathbf{2}, \boldsymbol{Ar} : \mathbf{2})$,

$$\Sigma \;=$$

where the only arrows are identity arrows. Then defining the constant diagram $\boldsymbol{D_1} := \boldsymbol{L_\Sigma} : \Sigma \mapsto \boldsymbol{L}$ and $\boldsymbol{D_2} := \boldsymbol{D} : \Sigma \to \boldsymbol{Set_*}$ as above. We extract the commutativity conditions that need to be satisfied by $\boldsymbol{\lambda}$:

$$\boldsymbol{K} := \left\{\boldsymbol{D(a) \circ \lambda_v = \lambda_w \circ L_\Sigma(a)} \;\middle|\; \text{For } \boldsymbol{a \in Ar}, \text{ where } \boldsymbol{v = o(a)}, \text{ and } \boldsymbol{w = e(a)}\right\}$$

Again, there are only two arrows corresponding to identities $\boldsymbol{Id_X}$ and $\boldsymbol{Id_Y}$ in the image of $\boldsymbol{D}$, both arrows map to $\boldsymbol{Id_L}$ in the image of $\boldsymbol{L_\Sigma}$, so we rewrite:

$$\boldsymbol{K = \{Id_X \circ p_1 = p_1 \circ Id_L,\; Id_Y \circ p_2 = p_2 \circ Id_L\}}$$

and these equalities hold because both images in each case are respectively: $\boldsymbol{(X, x_0)}$ and $\boldsymbol{(Y, y_0)}$. $\square$

This is very technical but easy to prove in the end, save the last version of $\boldsymbol{K}$ for easier arguments later!

- -

4a.) ii.) Now for the *terminal universal property.* Taking any other candidate cone of the same type, i.e. $\boldsymbol{(P, \pi_1, \pi_2)}$, we want to show:

$$\exists! \boldsymbol{f_P \in Hom_{Set_*}(P, L)}, \text{ such that } \boldsymbol{\pi_1 = p_1 \circ f_P} \text{ and } \boldsymbol{\pi_2 = p_2 \circ f_P}$$

Recall that $\boldsymbol{L := (X \times Y, (x_0, y_0))}$ and suppose $\boldsymbol{P := (P, p_0)}$, then define the map:

$$\boldsymbol{f_P : (P, p_0) \to (X \times Y, (x_0, y_0))}$$

via

$$\boldsymbol{f_P((p, p_0))} := \Big(\boldsymbol{(\pi_1(p),\; \pi_2(p)),\; (\pi_1(p_0), \pi_2(p_0))}\Big).$$

It follows that:

$$\boldsymbol{p_i \circ f_P((p, p_0)) = (\pi_i(p), \pi_i(p_0)) = \pi_i((p, p_0))}$$

so by arbitrariness of $\boldsymbol{(p, p_0) \in (P, p_0)}$, we have $\boldsymbol{\pi_i = p_i \circ f_P}$ as pointed-set morphisms.

Lastly, if $\boldsymbol{g_P : P \to L}$ were another such map making $\boldsymbol{\pi_i = p_i \circ g_P}$ then we'd have $\boldsymbol{p_i \circ f_P = p_i \circ g_P}$ for not just one, but (both) projections. Hence $f_P = g_P$. $\blacksquare$

(Continues)

4b.) Now we are talking about coproduct in $\boldsymbol{Set}_*$.

We wish to prove $(\widetilde{\boldsymbol{L}}, \widetilde{\boldsymbol{\lambda}}) := \Big(\big((\boldsymbol{X} \amalg \boldsymbol{Y})/\sim, [\boldsymbol{x}_0]\big),\ \boldsymbol{i}_1, \boldsymbol{i}_2 \Big)$ is the coproduct in $\boldsymbol{Set}_*$, where $\boldsymbol{i}_1 : (\boldsymbol{X}, \boldsymbol{x}_0) \to \big((\boldsymbol{X} \amalg \boldsymbol{Y})/\sim, [\boldsymbol{x}_0]\big)$ and $\boldsymbol{i}_1 : (\boldsymbol{Y}, \boldsymbol{y}_0) \to \big((\boldsymbol{X} \amalg \boldsymbol{Y})/\sim, [\boldsymbol{x}_0]\big)$ are inclusion maps that have been post-composed with the quotient map for the relation, $\boldsymbol{i}_1 := \boldsymbol{\pi} \circ \boldsymbol{\iota}_{\boldsymbol{X}}$ etc..

We skip showing candadacy as a *co-cone* of the appropriate type (Σ as before gives the type).

All the work we did in part (a) suggests that given another such candidate $(\boldsymbol{C}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$, we show existence and uniqueness of a pointed-set morphism from $\widetilde{\boldsymbol{f}}_C : \widetilde{\boldsymbol{L}} \to \boldsymbol{C}$ that makes $\boldsymbol{\mu}_j = \widetilde{\boldsymbol{f}}_C \circ \boldsymbol{i}_j$.

- -

Recall that $\widetilde{\boldsymbol{L}} := \big((\boldsymbol{X} \amalg \boldsymbol{Y})/\sim, [\boldsymbol{x}_0]\big)$ and suppose $\boldsymbol{C} := (\boldsymbol{C}, \boldsymbol{c}_0)$, then define the map:

$$\widetilde{\boldsymbol{f}}_C : \big((\boldsymbol{X} \amalg \boldsymbol{Y})/\sim, [\boldsymbol{x}_0]\big) \to (\boldsymbol{C}, \boldsymbol{c}_0)$$

via

$$\widetilde{\boldsymbol{f}}_C(([z], [\boldsymbol{x}_0])) := \begin{cases} \Big(\boldsymbol{\mu}_1\big(\boldsymbol{\pi}^{-1}([z])\big),\ \boldsymbol{c}_0\Big), & \text{if } \boldsymbol{\pi}^{-1}([z]) \in \boldsymbol{X} \\ \Big(\boldsymbol{\mu}_2\big(\boldsymbol{\pi}^{-1}([z])\big),\ \boldsymbol{c}_0\Big), & \text{otherwise} \end{cases}$$

where $\boldsymbol{\mu}_1 : \boldsymbol{X} \to \boldsymbol{C}$ and $\boldsymbol{\mu}_1 : \boldsymbol{x}_0 \mapsto \boldsymbol{c}_0$ and similarly $\boldsymbol{\mu}_2 : \boldsymbol{Y} \to \boldsymbol{C}$ and $\boldsymbol{\mu}_2 : \boldsymbol{y}_0 \mapsto \boldsymbol{c}_0$.

Also, $\boldsymbol{\pi}$ is the quotient map.

If we pre-compose with the i_j's we get:

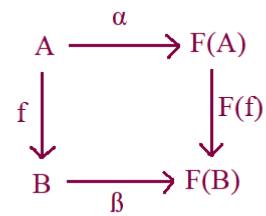
$$\widetilde{\boldsymbol{f}}_C \circ \boldsymbol{i}_1(\boldsymbol{x}, \boldsymbol{x}_0) = \big(\boldsymbol{\mu}_1(\boldsymbol{x}), \boldsymbol{c}_0\big) = \boldsymbol{\mu}_1((\boldsymbol{x}, \boldsymbol{x}_0)) \qquad \text{and} \qquad \widetilde{\boldsymbol{f}}_C \circ \boldsymbol{i}_2((\boldsymbol{y}, \boldsymbol{y}_0)) = \big(\boldsymbol{\mu}_2(\boldsymbol{y}), \boldsymbol{c}_0\big) = \boldsymbol{\mu}_2((\boldsymbol{y}, \boldsymbol{y}_0))$$

for each element in $(\boldsymbol{X}, \boldsymbol{x}_0)$ and $(\boldsymbol{Y}, \boldsymbol{y}_0)$ respectively. And hence by arbitrariness of the elements follows that the commutativity conditions are satisfied.

As before, uniqueness follows from the commutativity conditions. We equate across the $\boldsymbol{\mu}_j$'s and then we have equality of the morphisms over all elements in the quotient.

Final note, $\widetilde{\boldsymbol{f}}_C$ is a well-defined map since all equivalence classes besides $[\boldsymbol{x}_0]$ are singleton sets and even $([\boldsymbol{x}_0], [\boldsymbol{x}_0])$ and $([\boldsymbol{y}_0], [\boldsymbol{x}_0])$ both go to $(\boldsymbol{c}_0, \boldsymbol{c}_0)$. ■

Problem 5.) **Given a category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{C}$, a *coalgebra* of F is a pair (A, α) such that A is an object of $\mathcal{C}$ and $\alpha : A \to F(A)$ is a morphism in $\mathcal{C}$. The object A is called the *carrier* of the coalgebra (A, α). A morphism $(A, \alpha) \to (B, \beta)$ of coalgebras of F is a morphism $f : A \to B$ in $\mathcal{C}$ such that the following diagram commutes:**



(a) **Show that if A is a *carrier* of a coalgebra for F, then so is $F(A)$.**

(b) **Suppose (Z, ζ) is a *terminal object* in the category of coalgebras of F. Show that ζ is then an isomorphism.**

Proof:
We have for a given endo-functor $\boldsymbol{F : \mathcal{C} \to \mathcal{C}}$, the category:

$$\boldsymbol{CoAlg(F)} := \left\{ \boldsymbol{Obj(CoAlg(F))} := \{(\boldsymbol{A}, \boldsymbol{\alpha}) \mid \boldsymbol{A} \in \boldsymbol{Obj(\mathcal{C})}, \boldsymbol{\alpha} \in \boldsymbol{Hom(A, F(A))}\}, \right.$$
$$\left. \boldsymbol{Mor(CoAlg(F))} := \{\boldsymbol{f : A \to B} \mid \boldsymbol{F(f) \circ \alpha = \beta \circ f}\} \right\}$$

- -

a.) Suppose $\boldsymbol{A}$ is a carrier of a coalgebra. Then there exists a morphism $\boldsymbol{\alpha : A \to F(A)}$ making $(\boldsymbol{A, \alpha})$ an object in $\boldsymbol{CoAlg(F)}$.

Since $\boldsymbol{F}$ is assumed *covariant*, it follows then that $\boldsymbol{F(\alpha)} \in \boldsymbol{Hom}\big(\boldsymbol{F(A), F(F(A))}\big)$ and so $\big(\boldsymbol{F(A), F(\alpha)}\big) \in \boldsymbol{Obj(CoAlg(F))}$. Hence $\boldsymbol{F(A)}$ is also a carrier of a coalgebra. ■

---

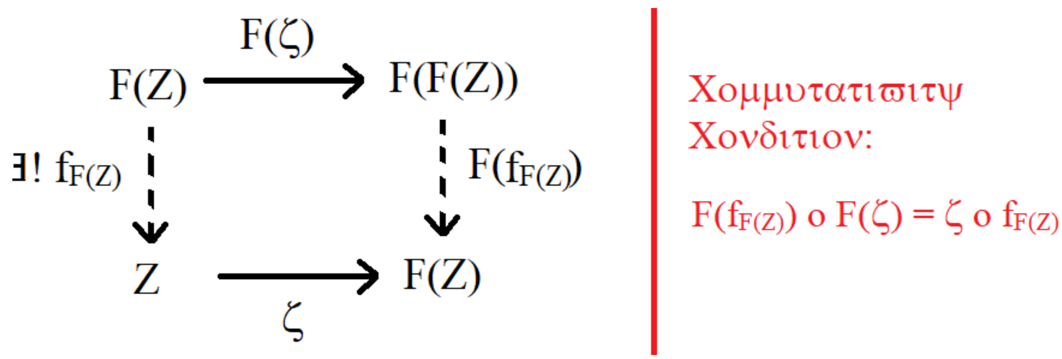
b.) Let $(\boldsymbol{Z, \zeta})$ be *terminal* in $\boldsymbol{CoAlg(F)}$. Then considering by part (a) that $\big(\boldsymbol{F(Z), F(\zeta)}\big)$ is also a coalgebra, we know there exists exactly one morphism: $\boldsymbol{f_{F(Z)} : \big(F(Z), F(\zeta)\big) \to (Z, \zeta)}$.

(Continues)

6b.) (Continued):
Opening up the definition of $\boldsymbol{CoAlg(F)}$ morphisms, we have there exists exactly one

$$\boldsymbol{f_{F(Z)} \in Hom_{\mathcal{C}}(F(Z), Z)}$$

making the diagram below commute:



From the commutativity condition above, together with the fact that our functor is *covariant*, we get:

$$\boldsymbol{F(f_{F(Z)} \circ \zeta) = \zeta \circ f_{F(Z)}} \qquad (\bigstar)$$

- -

Now, $\boldsymbol{f_{F(Z)} \circ \zeta \ \in Hom_{\mathcal{C}}(Z, Z)}$ and as well, $\boldsymbol{Id_Z \in Hom(Z, Z)}$ by axiom. By the terminal object property, we know there is only one morphism $\boldsymbol{f_Z : (Z, \zeta) \to (Z, \zeta) \in Mor(CoAlg(F))}$. So as coalgebra morphisms,

$$\boldsymbol{f_{F(Z)} \circ \zeta = Id_Z}$$

Combining this result with $\boldsymbol{(\bigstar)}$, we get:

$$\boldsymbol{\zeta \circ f_{F(Z)} = F(Id_Z) = Id_{F(Z)}} \qquad \text{(By functor axiom).}$$

Thus we have shown existence of a two sided inverse for $\boldsymbol{\zeta}$ - making it an *isomorphism*. ■

Problem 6.) **Let Grp denote the category of groups and let $F : Grp \to Grp$ send a group G to its opposite group G^{op} (i.e. the group whose underlying set is G, but with the operation given instead by $g * h := hg$, where the product hg is the usual product in G).**

(a) **For a group homomorphism $\varphi : G_1 \to G_2$, what is $F(\varphi)$? Show that F is a functor. Is it covariant or contravariant?**

(b) **Show that F is naturally isomorphic to the identity functor $Id_{Grp} : Grp \to Grp$. [Hint: you may want to first show that the map $G \to G^{op}$ that sends $g \mapsto g^{-1}$ is an isomorphism.]**

Proof:
a.) We wish to define the image of the morphism φ in either $\boldsymbol{Hom}(G_1^{op}, G_2^{op})$, pushing $\boldsymbol{F}$ towards covariance, or in $\boldsymbol{Hom}(G_2^{op}, G_1^{op})$, towards contra-variance.

But we have $\varphi(g * h) := \varphi(hg) = \varphi(h)\varphi(g) = \varphi(g) * \varphi(h)$. So $\varphi \in \boldsymbol{Hom}(G_1^{op}, G_2^{op})$.

So take $F(\varphi) := \varphi$.

To show $\boldsymbol{F}$ is a covariant functor then amounts to showing *preservation of composition and identities*.

Given $\varphi : G_1^{op} \to G_2^{op}$ and $\psi : G_2^{op} \to G_3^{op}$, we have by definition $F(\psi \circ \varphi) = \psi \circ \varphi = F(\psi) \circ F(\varphi)$ and $F(Id_G) = Id_{F(G)}$, since $F(G) = G^{op} = G$ as sets. ■

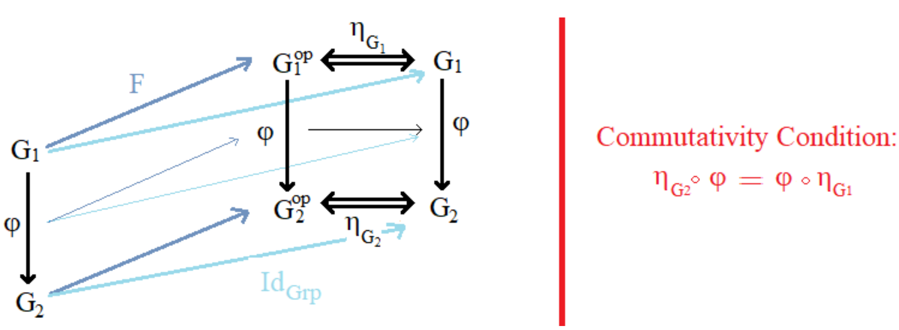
---

b.) Following the hint, WTS the maps $\eta_G : G \to G^{op}$ sending $g \mapsto g^{-1}$ are isomorphisms. But this is easy since inversion is involutive. And η_G is a homomorphism of groups since:

$$\eta_G(gh) = (gh)^{-1} = h^{-1}g^{-1} = \eta_G(g) * \eta_G(h) \qquad \text{and} \qquad \eta_G(1_G) = 1_G^{-1} = 1_G = 1_{G^{op}}$$

- -

This yields $\eta : F \xrightarrow{\cong} Id_{Grp}$, where $\eta = \{\eta_G\}_{G \in Grp}$ and for each morphism $\varphi : G_1 \to G_2$ the following holds:



since both sides of the commutativity conditions have image $\varphi(g)^{-1}$. ■

Problem 7.) **A morphism $\pi : A \to B$ in a category $\mathcal{C}$ is called an *epimorphism* if for every object C in $\mathcal{C}$, the induced function:**

$$\pi^* : Hom_{\mathcal{C}}(B, C) \to Hom_{\mathcal{C}}(A, C)$$

$$\varphi \mapsto \varphi \circ \pi$$

is injective. (This notion is meant to generalize that of a surjective function.)

(a) Show that in the category *Set* of sets, a morphism π is an epimorphism iff π is surjective.

(b) Show that for the category *CRing* of commutative rings (with identity, and with ring homomorphisms that preserve the identity), if R is an integral domain that is not a field and F is its field of fractions, then the natural map $R \to F$ is a non-surjective epimorphism.

>> Back to Section <<

Proof:
a.) (⇐) **:** Suppose we are in $\boldsymbol{Set}$. Let $\pi \in Hom(A, B)$ be a *surjective* set map and take two other maps $\varphi, \psi \in Hom(B, C)$ such that:

$$\varphi \circ \pi = \psi \circ \pi \qquad (\in Hom(A, C))$$

This means that:

$$\forall a \in A, \ \ \varphi \circ \pi(a) = \psi \circ \pi(a)$$

In particular, since $\forall b \in B, \exists a \in A$, such that $\pi(a) = b$, we can choose a *pre-image representative for each* b and throw away the other redundant equalities above to get:

$$\forall b_a \in B, \varphi(b_a) = \psi(b_a).$$

But clearly this says we have $\varphi = \psi$ as morphisms in $Hom(B, C)$. By arbitrariness of φ, ψ, we get that π^* is injective and hence π an epic. □

- -

(⇒) **:** Suppose now towards contradiction, that π is an epic but is not surjective. Then

$$\exists b_0 \in B, \ \forall a \in A, \ \ \pi(a) \neq b_0.$$

If we have φ, ψ as above, requiring that $\varphi \circ \pi = \psi \circ \pi$ is not enough to have $\varphi(b_0) = \psi(b_0)$, let alone to have equality on $B \backslash \pi(A)$. As we will show, there always exists two morphisms that differ on the complement of the image, but agree on the image $\pi(A)$:

$$\varphi(b) := \begin{cases} 1, & \text{if } b \notin \pi(A) \\ b, & \text{otherwise} \end{cases} \qquad \text{and} \qquad \psi(b) := \begin{cases} 0, & \text{if } b \notin \pi(A) \\ b, & \text{otherwise} \end{cases}$$

Not having total equality over B makes $\varphi \neq \psi$ and hence π^* is not injective. Contradiction. ■

(Continues)

7b.) Switch categories to $\boldsymbol{CRing}$ (commutative rings with $\mathbf{1}_{\boldsymbol{R}}$ and identity preserving ring hom's).

We have a ring $\boldsymbol{R}$ that is an *integral domain* but is *not a field*, so there are *no zero divisors* and *there exists elements that are not invertible under multiplication.*

[Recall: $\boldsymbol{F} = \{\frac{r}{s} \mid r \in R, s \in R - \{0\}\}$ together with fraction operations $+, *$ and identity elements. For more rigorous definition, see Dummit [?] Thm 15 (p.261). The fact that $\boldsymbol{R}$ is an integral domain is used to show the *divisors*, $\boldsymbol{s}$, are all nonzero elements here.]

The so called "natural map":

$$n : R \to F \qquad \text{via} \qquad n(r) := \frac{r}{1}$$

is actually a $\boldsymbol{CRing}$-morphism since $\mathbf{1} \mapsto \frac{1}{1}$ and $n(r * s + t) = \frac{r*s+t}{1} = \ldots = n(r) * n(s) + n(t)$.

WTS this map is a non-surjective, epimorphism.

- -

Note that since there exists non-invertible elements in $\boldsymbol{R}$, $\quad n : R \to F$ is *non-surjective* (we don't have a correspondent for some $\frac{1}{r_0}$ back in $\boldsymbol{R}$.)

Now, let $\varphi, \psi \in Hom(F, Z)$ for arbitrary $Z \in obj(CRng)$, be such that:

$$\varphi \circ n = \psi \circ n.$$

Then this gives:

$$\forall r \in R, \qquad \varphi(\frac{r}{1}) = \psi(\frac{r}{1}) \qquad \bigstar$$

But, φ and ψ both have domain $\boldsymbol{F}$, so we know the images of all fractions exist. So multiplying both sides of $\bigstar$ on the left (or right) by the element $\varphi(\frac{1}{r})\psi(\frac{1}{r})$ (assuming $r \neq 0$) and using axioms of *commutative ring homomorphisms* yields:

$$\forall r \in R - \{0\}, \qquad \psi(\frac{1}{r}) = \varphi(\frac{1}{r})$$

So we now have that $\varphi = \psi$, since they agree on all of $\boldsymbol{R}$ and all of $\frac{1}{R-\{0\}}$, they agree on all of $\boldsymbol{F}$–this can be seen by taking products $r * \frac{1}{s}$ and splitting the image of both maps via the ring homomorphism property and comparing results. So the natural map is an epic. ■

Problem 8.) **Let $\mathcal{C}$ be a category and let $f, g : X \to Y$ be two morphisms in $\mathcal{C}$. This is diagramatically written as:**

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

A *coequilizer* of f and g is an object Z of $\mathcal{C}$ equipped with a morphism $\pi : Y \to Z$ such that the diagram:

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\pi} Z$$

commutes (i.e. $\pi \circ f = \pi \circ g$) and satisfying the following *universal property*: for every morphism $\pi' : Y \to Z'$ with $\pi' \circ f = \pi' \circ g$, there exists a unique morphism $p : Z \to Z'$ such that the following diagram commutes:

$$\begin{array}{ccccc} X & \underset{g}{\overset{f}{\rightrightarrows}} & Y & \xrightarrow{\pi} & Z \\ & & & \searrow^{\pi'} & \downarrow{\scriptstyle \exists! p} \\ & & & & Z' \end{array}$$

(a) **Let Ab denote the category of abelian groups. Show that the coequilizer of two homomorphisms $f, g : A \to B$ in Ab is given by the quotient map:**

$$\pi : B \to B/Im(f - g)$$

(where $Im(f - g)$ denotes the image of $f - g$), i.e. the quotient map:

$$\pi : B \to coker(f - g)$$

(b) **Show that if $\pi : Y \to Z$ is the coequilizer of some pair $f, g : X \to Y$ in some category $\mathcal{C}$, then π is an epimorphism.**

The only reason abelian is necessary is so that the quotient is actually an object in the category.

Proof:

a.) It suffices to show $\left(B \backslash Im(f-g), \pi\right)$ satisfies the co-limit candidacy requirements (for type coequilizer) and satisfaction of the associated initial object universal property (we did a similar proof for products and coproducts before in Problem 4).

(Next Page)

Proof (Continued):

8a.)
Coequilizer Candidacy:
Recall $f, g : A \to B$ and we have the pair $\left(B \backslash Im(f-g),\ \pi : B \to B \backslash Im(f-g)\right)$ and we just need to show $\pi \circ f = \pi \circ g$ (as morphisms in Ab–i.e. as group homomorphisms).

Accordingly, fix $a \in A$, then:

$$\pi \circ f(a) = f(a) + Im(f-g)$$

$$= f(a) - (f(a) - g(a)) + Im(f-g) \qquad \text{[adding by an element of } Im(f-g)\text{]}$$

$$= g(a) + Im(f-g)$$

$$= \pi \circ g(a).$$

By arbitrariness of $a \in A$, the result follows. □

- -

Initial Object Universal Property:
Let $(C, \mu : B \to C)$ be another such candidate (i.e. such that $\mu \circ f = \mu \circ g$).

Then WTS there exists a unique morphism $\alpha_C : B \backslash Im(f-g) \to C$ with $\mu = \alpha_C \circ \pi$.
Define:

$$\alpha_C : B \backslash Im(f-g) \to C \qquad \text{via} \qquad \alpha_C(b + Im(f-g)) := \mu(b).$$

Clearly $\mu = \alpha_C \circ \pi$. And this map is actually well-defined since if we choose another coset representative:

$$\alpha_C\big(b + (f(x) - g(x)) + Im(f-g)\big) = \mu\big(b + f(x) - g(x)\big)$$

$$= \mu(b) + \mu(f(x)) - \mu(g(x)) = \mu(b)$$

since μ is a group homomorphism and by the candidacy commutativity condition.

Lastly, WTS uniqueness. If β_C is another map satisfying $\mu = \beta_C \circ \pi$, equating across μ then gives:

$$\alpha_C \circ \pi = \beta_C \circ \pi$$

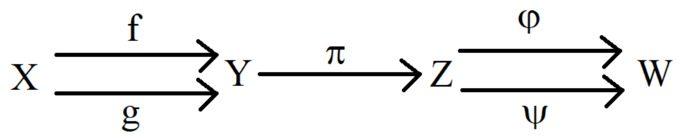
which gives $\alpha_C = \beta_C$ since quotient maps are surjective. ■

(Continues)

8b.) WTS if $\pi : Y \to Z$ is the coequilizer of two morphisms $f, g : X \to Y$ in an arbitrary category, then π is an epimorphism.

- -

Consider the following figure for arbitrary $W \in obj(\mathcal{C})$:



Suppose $\varphi, \psi \in Hom(Z, W)$ are such that $\varphi \circ \pi = \psi \circ \pi$.

Then we have:

$$(\varphi \circ \pi) \circ f = (\varphi \circ \pi) \circ g \qquad \text{and} \qquad (\psi \circ \pi) \circ f = (\psi \circ \pi) \circ g$$

due to *associativity* and the fact that π gives us $\pi \circ f = \pi \circ g$.

This says $\big(W, \varphi \circ \pi\big)$ and $\big(W, \psi \circ \pi\big)$ are candidates for the coequilizer-type colimit. By the *(initial object) universal property* that $\big(Z, \pi\big)$ has then, we know there exists exactly one morphism coming out of the coequilizer and pointing to each of the candidates, factoring each of their morphisms.

More formally, we have:

$$\exists! \ \alpha_{\varphi\circ\pi} \in Hom(Z, W) \qquad \text{such that} \qquad (\varphi \circ \pi) = \alpha_{\varphi\circ\pi} \circ \pi$$

and

$$\exists! \ \alpha_{\psi\circ\pi} \in Hom(Z, W) \qquad \text{such that} \qquad (\psi \circ \pi) = \alpha_{\psi\circ\pi} \circ \pi$$

Reiterating, if we look at the collection:

$$\{\beta \in Hom(Z, W) \mid \varphi \circ \pi = \beta \circ \pi\},$$

we know by hypothesis that φ and ψ are in this collection and so by uniqueness they must be the same. That is, $\varphi = \psi$ and by arbitrariness of these morphisms and W we have that π is an epimorphism. ■

Problem 9.) **Let $F : Grp \to Set$ be the forgetful functor sending a group to its underlying set. Let $G : Grp \to Set$ be the functor given by**

$$X \mapsto Hom_{Grp}(\mathbb{Z}, X)$$

sending a group X to the set of group homomorphisms from the additive group of integers to X.

a.) **Show that G is indeed a covariant functor.**

b.) **Show that F and G are naturally isomorphic (i.e. show that F is represented by $\mathbb{Z}$.**

Proof:
a.) G is nothing more than the covariant hom functor $H^{\mathbb{Z}}$ indexed by $\mathbb{Z} \in Obj(Grp)$. We've proven this functor is covariant in Problem 1. ■

---

b.) Recall:

$$F : Grp \to Set; \quad \text{and} \quad G : Grp \to Set;$$
$$X \mapsto |X| \qquad\qquad X \mapsto Hom(\mathbb{Z}, X)$$

We wish to show there exists a *natural isomorphism* between these two functors.

Accordingly, define:

$$\eta : F \to G;$$
$$\left\{ \eta_X : |X| \to Hom(\mathbb{Z}, X); \quad x \mapsto (f_x : 1 \mapsto x) \right\}_{X \in Obj(Grp)}$$

- -

We need to show (i) this map satisfies the proper commutativity conditions and (ii) that each η_X is an isomorphism (in Set).

i.) Let $X, Y \in Obj(Grp)$ and $f : X \to Y$ be arbitrary. WTS

$$G(f) \circ \eta_X = \eta_Y \circ F(f) \qquad \in Hom_{Set}(|X|, Hom_{Grp}(\mathbb{Z}, Y))$$

We have for a given $x \in |X|$ that: $G(f) \circ n_X(x) := G(f) \circ (f_x : 1 \mapsto x) := f \circ f_x$.

As well, $\eta_Y \circ F(f)(x) := \eta_Y \circ f(x) := f_{f(x)} : 1 \to f(x) = f \circ f_x$.

Arbitrariness of x, X, Y, f finishes this. So we indeed have a natural transformation. □

- -

(Continues)

9b.) (Continued):

ii.)
Injective:

Suppose for arbitrary $\boldsymbol{X} \in \boldsymbol{Obj}(\boldsymbol{Grp})$ and $\boldsymbol{x}, \boldsymbol{y} \in |\boldsymbol{X}|$ are such that $\boldsymbol{\eta_X}(\boldsymbol{x}) = \boldsymbol{\eta_X}(\boldsymbol{y})$. Then:

$$(\boldsymbol{f_x} : \mathbf{1} \mapsto \boldsymbol{x}) \equiv (\boldsymbol{f_y} : \mathbf{1} \mapsto \boldsymbol{y})$$

And in particular this says $\boldsymbol{x} = \boldsymbol{f_x}(\mathbf{1}) = \boldsymbol{f_y}(\mathbf{1}) = \boldsymbol{y}$, so that $\boldsymbol{x} = \boldsymbol{y}$. $\square$

Surjective:

For any given $\boldsymbol{g} \in \boldsymbol{Hom}(\mathbb{Z}, \boldsymbol{X})$, there exists $\boldsymbol{x} := \boldsymbol{g}(\mathbf{1}) \in |\boldsymbol{X}|$ such that

$$\boldsymbol{\eta_X}(\boldsymbol{x}) = (\boldsymbol{f_{g(1)}} : \mathbf{1} \mapsto \boldsymbol{g}(\mathbf{1})) \equiv \boldsymbol{g}$$

$\therefore$ We have shown $\boldsymbol{F}$ is isomorphic to a covariant hom functor indexed by $\mathbb{Z}$, therefore the forgetful functor, $\boldsymbol{F}$, has representing object $\mathbb{Z}$. $\blacksquare$

2. Advanced Material

Subsection Contents:

>> Jump to References

1: Some Tools for Homological Algebra:
In this section, we aim to develop the definition of an n^{th}*-homology object*. Such objects are certain *quotients* sitting inside the nodes of a *chain complex*. If a homology object is zero, the sequence is exact at the node it lives in. It is of course the nonzero homology objects we find interesting. Very briefly, given a base category, one assigns chain complexes to its objects and the resulting homology sequences characterize those objects [speculative statement]. For example, in [?], topological spaces with a cover are assigned sheaves of abelian groups, whereby cochains, cocycles, and coboundaries are developed in the nodes of sequences. And this yields the cohomology of the topological space. None of the references, nor wikipedia specifies this connection, it seems diagram chasing is the more important aspect. Either way...

On (p.111 [?]), Schubert states: "Abelian categories are the proper framework for the study of exact sequences. They are the foundation of homological algebra... $\boldsymbol{Ab}$, $_{\boldsymbol{R}}\boldsymbol{Mod}$, $\boldsymbol{Mod_R}$ are examples of abelian categories."

Switching to Mac Lane's text now...

• Def: (p.28 [?]) An **Ab-category** (also called **preadditive**), is one such that:

i.) $\forall \boldsymbol{A, B}, \exists +_{\boldsymbol{AB}}$ and $\boldsymbol{0_{AB} : A \to B}$ making $\left\{\boldsymbol{Hom(A,B), +_{AB}, 0_{AB}}\right\}$ an *abelian group*.

ii.) We require that the composition in $\boldsymbol{\mathcal{C}}$ is "bilinear" with respect to each $+_{\boldsymbol{AB}}$. That is,

$$\forall \boldsymbol{f, g \in Hom(A,B), \forall h \in Hom(B,C)}, \qquad \boldsymbol{h \circ (f+g) = (h \circ f) + (h \circ g)} \text{ and}$$

$$\forall \boldsymbol{f, g \in Hom(B,C), \forall h \in Hom(A,B)}, \qquad \boldsymbol{(f+g) \circ h = (f \circ h) + (g \circ h)}.$$

• Def: (p.194 [?]) Given two objects $\boldsymbol{A}$ and $\boldsymbol{B}$ in an Ab-category, a **biproduct** object, denoted $\boldsymbol{A \oplus B}$, is simultaneously a *product* and a *co-product* with the compatibility conditions:

i.) $\boldsymbol{p_1 \circ i_1 = Id_A}$, $\quad \boldsymbol{p_2 \circ i_2 = Id_B}$, $\quad$ and

ii.) $\boldsymbol{i_1 \circ p_1 = Id_{A \oplus B}}$, $\quad \boldsymbol{i_2 \circ p_2 = Id_{A \oplus B}}$.

• Def: (p.196 [?]) An **additive category** is an Ab-category which has a zero object $\boldsymbol{0 \in Obj(\mathcal{C})}$ (not just zero morphisms $\boldsymbol{0_{AB} : A \to B}$ in each Hom collection), and a biproduct, $\boldsymbol{A \oplus B}$ for each pair of objects.

(Continues)

• <u>Def:</u> (p.197 [?]) If $\mathcal{C}$ and $\mathcal{D}$ are Ab-categories, an **additive functor**, $F : \mathcal{C} \to \mathcal{D}$, is one such that:

$$\forall A, B,\ \forall f, g \in Hom(A, B), \qquad F(f + g) = F(f) + F(g),$$

where of course, the addition structures are in $Hom_{\mathcal{C}}(A, B)$ and $Hom_{\mathcal{D}}(F(A), F(B))$ respectively.

• <u>Def:</u> (p.198 [?]) An **abelian category** is an additive category such that:

i.) Every morphism $f : A \to B$ has a *kernel* and a *cokernel* and

ii.) Every monic is a kernel and every epic is a cokernel.

We have now set the stage. Enter sequences and exactness definitions...

• <u>Def:</u> (Prop 1, p.199-200 [?]) Given a morphism $f : A \to B$ in an abelian category, define:

$$im(f) := ker(coker(f)) \qquad \text{and} \qquad coim(f) := coker(ker(f)).$$

Then, up to isomorphism, we have the unique factorization of any morphism given by:

$$f = im(f) \circ coim(f). \ \blacksquare$$

• <u>Def:</u> (p.200 [?]) A composable pair of morphisms:

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is **exact at B** when $im(f) \equiv ker(g)$ (as subobjects of B).

• <u>Def:</u> An **exact sequence** in $\mathcal{C}$ is a sequence of composable morphisms:

$$\ldots \to A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \to \ldots$$

that is exact at every object A_n. In particular, a **short exact sequence** is an exact sequence of the form:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

where $\mathbf{0}$ is the zero object in our abelian category.

(Continues)

• Def: (p.201 [?]) An **exact functor**, $F : \mathcal{C} \to \mathcal{D}$ is one that preserves all finite limits and all finite colimits (in particular it preservers kernels and cokernels):

$$F\big(ker(f)\big) = ker\big(F(f)\big) \quad \text{and} \quad F\big(coker(f)\big) = coker\big(F(f)\big).$$

It hence preserves images, coimages, and carries exact sequences to exact sequences [Exercise]. The weaker notions of **left-exact** and **right-exact** functors are defined as preserving only finite limits OR finite colimits respectively.

Now, for the good part.

• Def: (p.202 [?]) In an abelian category $\mathcal{C}$, a **chain complex** is a sequence of composable morphisms in $\mathcal{C}$, more traditionally denoted:

$$\ldots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \ldots$$

such that $\partial_n \circ \partial_{n+1} = 0_{C_{n+1}C_n}$ for every n.

• Def (My Own): Suppose $f : A \to X$ and $g : B \to X$ are given such that $f \leq g$. We wish g/f to be a quotient object of B, but it needs to be explicitly in terms of f and g. So let us define:

$$g/f := coker(\varphi) : B \to \widetilde{K}_\varphi.$$

where $\varphi : dom(f) \to dom(g)$ is the unique morphism such that $f = g \circ \varphi$. Let this specify up to equivalence class the epic obtained. *Note: In other theories (abelian groups etc.), cokernel is given by* $\widetilde{K}_\varphi := B/\varphi(A)$.

• Def: (p.202 [?]) In the context of a chain complex, we define the **n^{th}-homology object** by:

$$H_n C := ker(\partial_n)/im(\partial_{n+1}),$$

where the symbology on the right hand side denotes a quotient of subobjects of C_n.

- -

Elaborating:

$$ker(\partial_n) : K_{\partial_n} \to C_n \quad \text{and} \quad im(\partial_{n+1}) := ker(coker(\partial_{n+1})) : K_{coker(\partial_{n+1})} \to C_n$$

and since kernels are monics, each gives rise to a subobject (passing to the equivalence classes). [Exercise: Show $im(\partial_{n+1}) \leq ker(\partial_n)$.] Now apply the definition of the quotient of subobjects.

Lastly, I want to include one more definition:

• Def: (p.202 [?]) We can define a **morphism of short exact sequences** as a family of morphisms in $\mathcal{C}$ between corresponding nodes of the sequences such that the resulting diagram between them commutes. The collection of all short exact sequences in a category, together with all such morphisms and the induced composition defines a category we call $Ses(\mathcal{C})$.

[3:13 am. 10.17.2020. KTS]

References

[1] Baez, J. (1997). *An Introduction to n-Categories*. Dept. of Mathematics, UC Riverside. Retrieved from URL:
<< https://arxiv.org/pdf/q-alg/9705009.pdf >> Accessed June 23, 2020.

[2] Denecke, K., & Wismath, S. L. (2002). *Universal Algebra and Applications in Theoretical Computer Science*. Chapman & Hall/CRC.

[3] Dummit, D. & Foote, R. (2004). *Abstract Algebra*. John Wiley & Sons.

[4] Hungerford, T. W. (1974). *Algebra*. New York: Springer.

[5] Kashiwara, M. & Schapira, P. (2010). *Categories and Sheaves*. Berlin: Springer.

[6] Milewski, B. (2017, Feb 23). Category Theory II 1.2: Limits [Video File]. Retrieved from URL:
<< https://www.youtube.com/watch?v=sx8FELiIPg8 >> Accessed June 23, 2020.

[7] Miranda, R. (1995). *Algebraic Curves and Riemann Surfaces*. Springer.

[8] Rotman, J. (2003). *Modern Algebra*. Prentice Hall.

[9] Smith, K. (2020). *The Monodromy Representation*. 545 CSULB.

[10] Smith, K. (2020). *Čech Cohomology Groups and Riemann Surfaces*. 566 CSULB.

[11] Smith, K. (2020). *Modal Logics in Alg(F) and CoAlg(F)*. 697 CSULB.

[12] Mac Lane, S. (2000). *Categories for the Working Mathematician*. Springer

[13] Schubert, H. (1972). *Categories*. Berlin: Springer.

[14] Wikipedia contributors. Glossary of category theory. Wikipedia, The Free Encyclopedia. May 20, 2020, 08:25 UTC. Available at:
<< https://en.wikipedia.org/w/index.php?title=Glossary_of_category_theory&oldid=957733229 >> Accessed June 1, 2020.

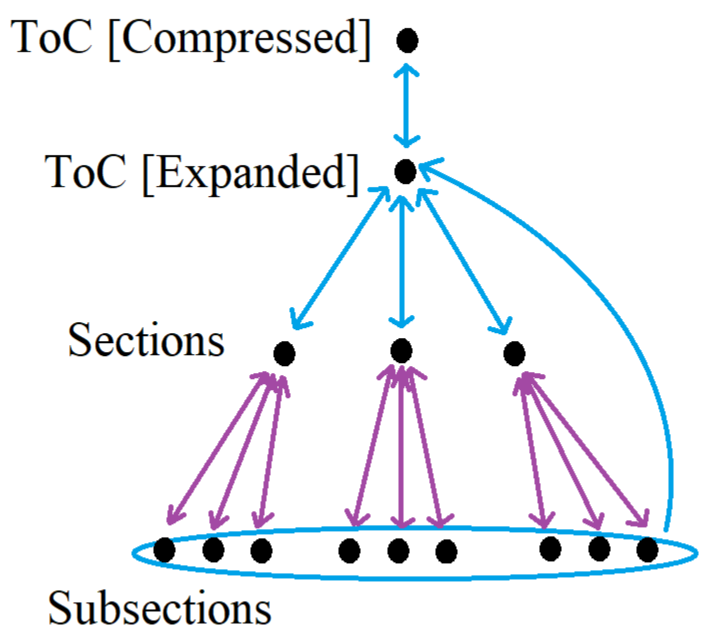
[15] Wikipedia contributors. Category theory. Wikipedia, The Free Encyclopedia. May 18, 2020, 10:14 UTC. Available at:
<< https://en.wikipedia.org/w/index.php?title=Category_theory&oldid=957337856 >>
Accessed June 1, 2020

[16] Wikipedia contributors. Limit (category theory). Wikipedia, The Free Encyclopedia. April 20, 2020, 15:49 UTC. Available at:
<< https://en.wikipedia.org/w/index.php?title=Limit_(category_theory)&oldid=952102587 >>
Accessed July 9, 2020.

[17] Moschovakis, Y. (2006). *Notes on Set Theory*. Springer

Using this PDF:

This text was designed with a hyperlink navigational structure indicated by the following diagram:



with downward arrows usually being labeled by name and any upward arrows being given by the "/ << /" or "/ < /" icons in the bottom left corner of each page. There may also be internal links not specified here. If you keep this in mind while reading, it makes accessing information in the document easy. Press back to get started.