♦ The Monodromy Representation ♦

A Math 545 Project at CSULB, Spring 2020

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Contents:

1: Introduction

In this work, we develop a particular construction, relating subgroups of the fundamental group of a “nice” space to that of permutation matrices on a vector space with dimension corresponding to that of the degree of the associated connected covering space. By nice, we mean of course *connected, locally path-connected, and semi-locally simply connected*.

Section 2 proceeds to define most of the necessary terminology from algebraic topology. Section 3 defines group actions and the action of the fundamental group on the universal covering space, and then lists an important relationship between classes of covering spaces and conjugacy classes of subgroups of the fundamental group. Section 4 builds on this with a quick discussion and delivers the final definition of the monodromy representation.

Section References:

Each section was heavily inspired (if not transcribed at points) by the texts listed in the bibliography. For the interested reader, these are the page ranges we used, more direct citations can be found throughout.

$>$ 2: Miranda (pg.84-85) and Hatcher (p.25-78).

$>$ 3: Miranda (pg.75-85), Hatcher (p.28-70), Dummit (p.574), and Denecke (p.40).

$>$ 4: Miranda (86-87).

2: Preliminaries in Algebraic Topology

<< 2.1: The Fundamental Group of a Topological Space >>

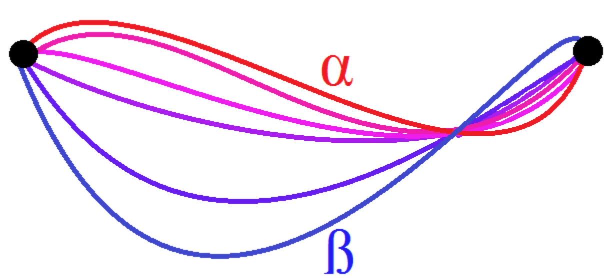
• Def: A (finite) **path** in a topological space $\boldsymbol{X}$ is a map: $\boldsymbol{\alpha : [0,1] \to X}$.
We call $\boldsymbol{\alpha(0)}$ and $\boldsymbol{\alpha(1)}$ the **endpoints of** $\boldsymbol{\alpha}$. In particular, when the endpoints are the same, we call α a **loop in** $\boldsymbol{X}$ and usually use the symbol '$\boldsymbol{\gamma}$' instead.

• Def: Given two paths $\boldsymbol{\alpha, \beta : [0,1] \rightrightarrows X}$ with the same starting and ending points, we define a *linear* **homotopy from** $\boldsymbol{\alpha}$ **to** $\boldsymbol{\beta}$ relative to fixed endpoints $\boldsymbol{\alpha(0)}$ and $\boldsymbol{\alpha(1)}$ by a map:

$$\boldsymbol{H : [0,1] \times [0,1] \to X}; \qquad \boldsymbol{H_s(t) \equiv H(s,t) := \alpha(t) + (\beta(t) - \alpha(t)) \cdot s}.$$

More generally, **homotopies** between paths are just *continuous deformations* of one into the other (end points not necessarily fixed). This can be upgraded to *homotopies of maps of topological spaces*, but we don't need that much here.

Informally, $\boldsymbol{t}$ is the *time* or *arclength* parameter and $\boldsymbol{s}$ is the *family* or *transverse* parameter. Notice that above, when $s = 0$ and $s = 1$ respectively, we have the family members $\alpha(t)$ and $\beta(t)$.

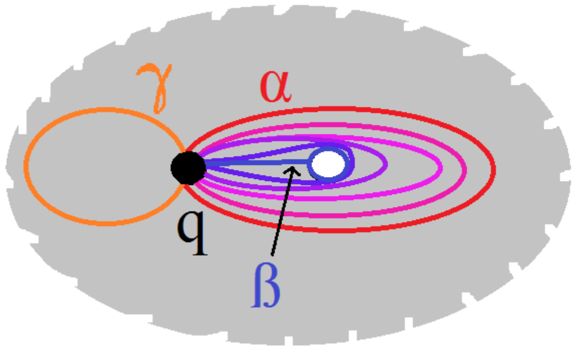


• Def: We can define a relation '$\sim$' by existence of homotopies between paths:

$\boldsymbol{(\alpha \sim \beta) \leftrightarrow \exists H(s,t)}$ (a homotopy between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ relative to fixed endpoints)

This relation turns out to be an equivalence relation on the set of all paths in X. Hence it partitions $\{$paths in $X\}$ into equivalence classes, each of which we denote by say $[\boldsymbol{\alpha}]$.

• <u>Defs:</u> We are concerned with encoding "holes" in the topological space into algebraic quantities, one of the ways this is accomplished is by looking at equivalence classes of loops.



For example, α and β in this figure can be deformed into eachother without the hole in the space creating a discontinuity for the homotopy map; γ is so called **contractible** since it can be deformed to the **constant loop**. Since neither α nor β are contractible, they are inherently members of a different homotopy class than γ.

Note however that the combined path going once around γ at double the speed and then once around α at double the speed is a path in the same equivalence class of as α. This is known as the **path product:** $\boldsymbol{\gamma \cdot \alpha}$ (listed contrary to our composition of functions intuition).

The path product in general is a piecewise-defined path, where each portion is a re-parameterized version of one of the constituent paths, meant to satisfy the original definition we gave over the interval $[0, 1]$.

We define the **reverse path**, denoted $\boldsymbol{-\alpha}$ or $\boldsymbol{\bar{\alpha}}$ as the original path, precomposed with a *time-reversal homeomorphism*, say $\boldsymbol{\varphi(t) = 1 - t}$.

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• <u>Def:</u> It can be shown (pg.26 [1]) that the set of equivalence classes of loops base-pointed at $q \in X$ forms a *group* structure with path product being upgraded to the operation, i.e.

$$[\boldsymbol{\alpha}] * [\boldsymbol{\beta}] := [\boldsymbol{\alpha} \cdot \boldsymbol{\beta}], \qquad \text{(well-defined and associative)},$$

the identity is the constant loop class $\mathbb{1} := [\boldsymbol{q}]$ of the basepoint, and we get inverses of elements via the reverse path classes $[\boldsymbol{\alpha}]^{-1} := [-\boldsymbol{\alpha}]$. This group is called the (first) **fundamental group of $\boldsymbol{X}$ at $\boldsymbol{q}$** and is denoted $\boldsymbol{\pi_1(X, q)}$. There are n^{th} fundamental groups as well, we don't use them here.

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<< 2.2: Some Spatial Properties >>

The following properties are important for developing the universal cover in the next section.

• Def: A topological space X is **connected** if it is not the disjoint union of two other topological spaces. Within a single topology, a connected subset is one such that it is not a disjoint union of two subspaces of X.

• Def: A topological space X is **path-connected** if for any two points in X, there exists a continuous path between them. A space is **locally path-connected** if for every point $p \in X$, there exists a neighborhood N_p such that N_p is path-connected.

• Def: **Simply connected spaces** are topological spaces with trivial fundamental group, i.e. $\pi_1(X, q) \cong \{e\}$ (there are no holes so every loop is contractible).

• Def: For a continuous map of pointed-topological spaces $\varphi : (X, q) \to (Y, \varphi(q))$, we define an **induced map**:

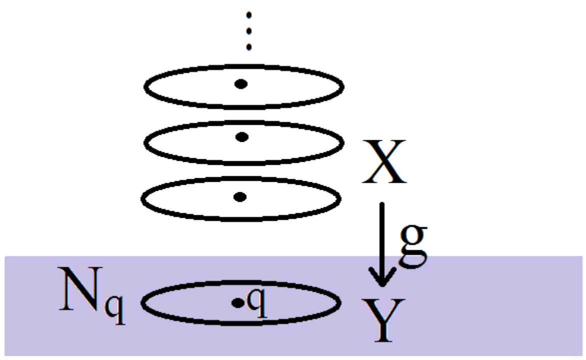
$$\boldsymbol{\varphi_* : \pi_1(X, q) \to \pi_1(Y, \varphi(q)); \quad \varphi([\gamma]) := [\varphi(\gamma)].}$$

• Def: **Semilocally simply connected spaces** are such that for each point $p \in X$, there exists a neighborhood N_p such that the *inclusion-induced map* $\pi_1(N_p, p) \to \pi_1(X, p)$ is trivial (pg.63 [1]).

<< 2.3: Topological Covering Spaces, TopCov(Y) >>

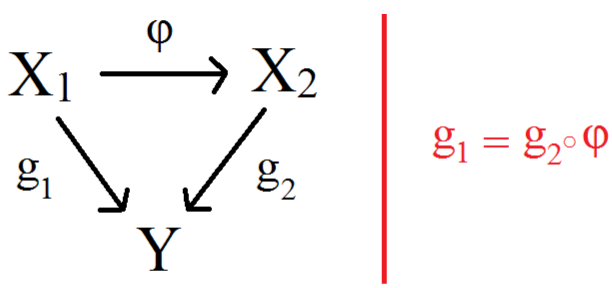
• Def: Given a topological space $\boldsymbol{Y}$, a map $\boldsymbol{g : X \to Y}$ is called a **covering map**, with **base space** $\boldsymbol{Y}$ and **total space** $\boldsymbol{X}$ if $\forall q \in Y$, there exists a neighborhood $N_q \subseteq Y$ such that the pre-image $g^{-1}(N_q)$ of the neighborhood is a disjoint union of subsets of X each of which maps homeomorphically to Y via g.

Covering spaces are jokingly thought of as stacks of pancakes hovering above another pancake. This is of course a loose interpretation of covering spaces viewed only as restricted to the fiber "above" a disk.



A more concrete example of a covering space is given by $\mathbb{R} \to \mathbb{S}^1$ or by $\mathbb{C} \to \mathbb{C}/L$ (reals covering the circle or the complex plane covering the torus in identification space).

• Time to define the morphisms. Given two covering spaces over the same base space, namely $\boldsymbol{g_1 : X_1 \to Y}$ and $\boldsymbol{g_2 : X_2 \to Y}$, a **covering space morphism** is just a continuous map $\boldsymbol{\varphi : X_1 \to X_2}$ making the following diagram commute:



In the event that φ is a *homeomorphism*, we say φ is a **covering space isomorphism** or **deck transformation**. The latter term appealing to the notion of shuffling the hovering pancakes.

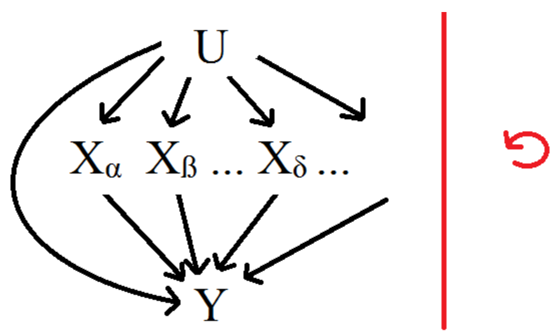
• <u>Def:</u> With objects as covering spaces ($g : X \to Y$) and morphisms as covering morphisms (φ above), together with the identity morphisms and composition of continuous maps, we obtain a category called $\boldsymbol{TopCov(Y)}$ (interested reader can check).

In categorical language, for certain special topological spaces Y, we have existence of *initial objects* $\boldsymbol{f : U \to Y}$ in $\boldsymbol{TopCov(Y)}$. That is, $f : U \to Y$ has the property that every other object in the category has in incoming morphism from it.

Bringing the discussion back to algebraic topology proper:

• <u>Def/Prop:</u> (See pg.68 [1]) For a connected, locally path-connected and semilocally simply connected topological space Y, there exists a simply connected covering space $\boldsymbol{f : U \to Y}$, called the **universal covering space of** $\boldsymbol{Y}$. This covering space satisfies the following *universal property*:

For every other covering space $g : X \to Y$, there exists a covering space morphism $\varphi : U \to X$ (also a covering space) such that $f = g \circ \varphi$. This property makes it unique up to covering space isomorphism (as the reader can check).



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Before we move on to getting at the main attraction, we mention two more things that only really fit here:

> If the base space has additional, local properties, they usually get inherited in the total space via the fact that covers are local homeomorphisms. (See wikipedia's "Covering Space" article [5]). For example, a Riemann Surface lifts to a R.S.

> **Path-Lifting Property** (p.60 [1]): Given a point and a loop in the base space, there exists a unique path starting at each point in the fiber.

3: Acting Fundamental Groups on Universal Covers

<< 3.1: Group Actions >>

• Def: Let $\boldsymbol{G}$ be a group and $\boldsymbol{X}$ a topological space. We define an **action of $\boldsymbol{G}$ on $\boldsymbol{X}$** to be a map:

$$\boldsymbol{\theta : G \times X \to X} \qquad \text{such that:}$$

1.) $\forall g, h \in G, \forall p \in X$ we have $\theta(gh, p) = \theta(g, \theta(h, p))$ and
2.) $\forall p \in X$ we have $\theta(e, p) = p$, where e is the identity of the group.

We also denote $\boldsymbol{g * p := \theta(g, p)}$.

• Def: Given such an action and a point $p \in X$, we define the **$\boldsymbol{G}$-orbit of $\boldsymbol{p}$** or simply the *orbit of* p to be the set:

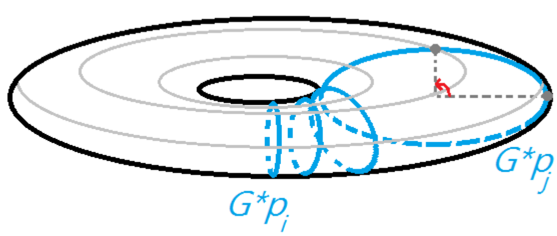
$$\boldsymbol{G * p = \{g * p \mid g \in G\}},$$

that is, the collection of images of p with respect to the action of different group elements.

• Def: We can define a relation, $\sim$, on X via orbit inclusion. That is,

$$\boldsymbol{\forall p, q \in X,\ (p \sim q) \leftrightarrow (p \in G * q)}.$$

This turns out to be an equivalence relation and hence partitions the space into orbits. We call $\boldsymbol{X/\sim}$ the **orbit space with respect to $\boldsymbol{G}$**. Alternatively, this is denoted $\boldsymbol{X/G}$.



We can give X/G the *quotient topology* (and many other properties X has) via the **natural quotient map $\boldsymbol{\pi : X \to X/G}$** (see p.75 Miranda).

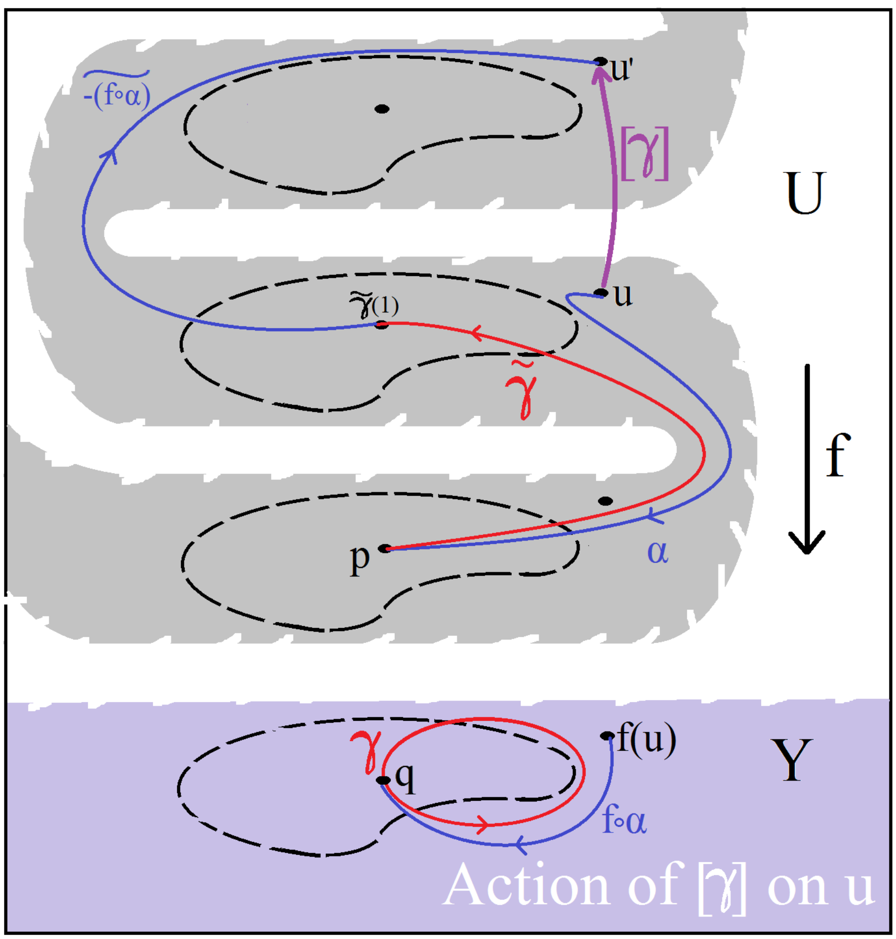
<< 3.2: On Universal Covers >>

Let $\boldsymbol{f : U \to Y}$ be the universal covering space for $\boldsymbol{Y}$ and let $\boldsymbol{G := \pi_1(Y, q)}$ at some basepoint $\boldsymbol{q} \in \boldsymbol{Y}$. Then, selecting a pre-image $\boldsymbol{p} \in \boldsymbol{f^{-1}(q)}$, we obtain an action $\boldsymbol{G} \times \boldsymbol{U} \to \boldsymbol{U}$ via:

$$(\bigstar) \qquad [\gamma] * \boldsymbol{u} := -(\widetilde{\boldsymbol{f} \circ \boldsymbol{\alpha}_{u,p}})(\boldsymbol{1})$$

where $\boldsymbol{\alpha}$ is a path from u to p in U, $-(f \circ \alpha)$ is a reverse path in the base space from $f(p)$ to $f(u)$, and $\widetilde{-(\boldsymbol{f} \circ \boldsymbol{\alpha})}$ is the unique lift of $-(f \circ \alpha)$ starting at $\widetilde{\gamma}(1)$, which is the endpoint of the unique lift of γ starting at p. See the figure below.

Note especially that $u' \in f^{-1}(f(u))$, so that the fibers are preserved under the action.



<< 3.3: A Galois Correspondence >>

The action of the fundamental group on its space's universal cover described above gives rise to a type of *Galois correspondence* between:

$$\left\{\begin{matrix}\text{isomorphism classes of}\\ \text{path-connected coverings}\\ \boldsymbol{g : X \to Y}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{conjugacy classes}\\ \text{of subgroups}\\ \boldsymbol{H \leq \pi_1(Y, q)}\end{matrix}\right\},$$

in the case of a *path connected, locally path-connected, and semilocally simply connected* base space Y (when the base points are ignored in the isomorphisms) [See (Theorem 1.38, p.67 [1]) then see (pg.85 [4])].

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There is too much to prove here for where we are trying to go with the theory, so we defer the proof of the correspondence to the references (note that induced maps are used). However, there are a couple things worth mentioning:

1.) The reason for the terminology used above is the way in which the correspondence happens. It resembles the way subfields of splitting fields of polynomials correspond to the subgroups of the poly's automorphism group (see pg.574 Dummit [2]), in particular note the *inclusion reversal.*

Consider the action of the *trivial subgroup* of $\pi_1(Y, q)$ on the universal cover $f : U \to Y$. There is one element $[\textit{constant loop}] \in H \leq \pi_1(Y, q)$ and as one can see in the above figure, this makes the action fix every point in U since the lift of the reversed path $-f \circ \alpha$ starts at p and ends back at u. Hence the orbits are all just singleton sets, i.e. the covering space corresponds to $U/[c] \equiv U \to Y$.

On the other extreme, considering instead the action of the entire group yields the orbits to be the entire fiber above each point in Y, so we actually get the orbit space is homeomorphic to Y. So the covering space looks like $Y \to Y$.

Lastly, there is a precise notion of *Galois connection* given in (pg.40 of [3]) if one wanted to tie these ideas together better!

2.) More important for our needs is a result of this correspondence:

(Continues)

• <u>Prop:</u> Let $g : X \to Y$ be a covering space such that Y is "nice enough" and $H \leq \pi_1(Y, q) =: G$ its corresponding subgroup, then:

$$\boldsymbol{deg(g) = [G : H]}.$$

*Here, **degree** is the number of preimages of g at any given point in Y.*

<u>Proof:</u> [See Prop 1.3.2 (p.61) in [1]] ■

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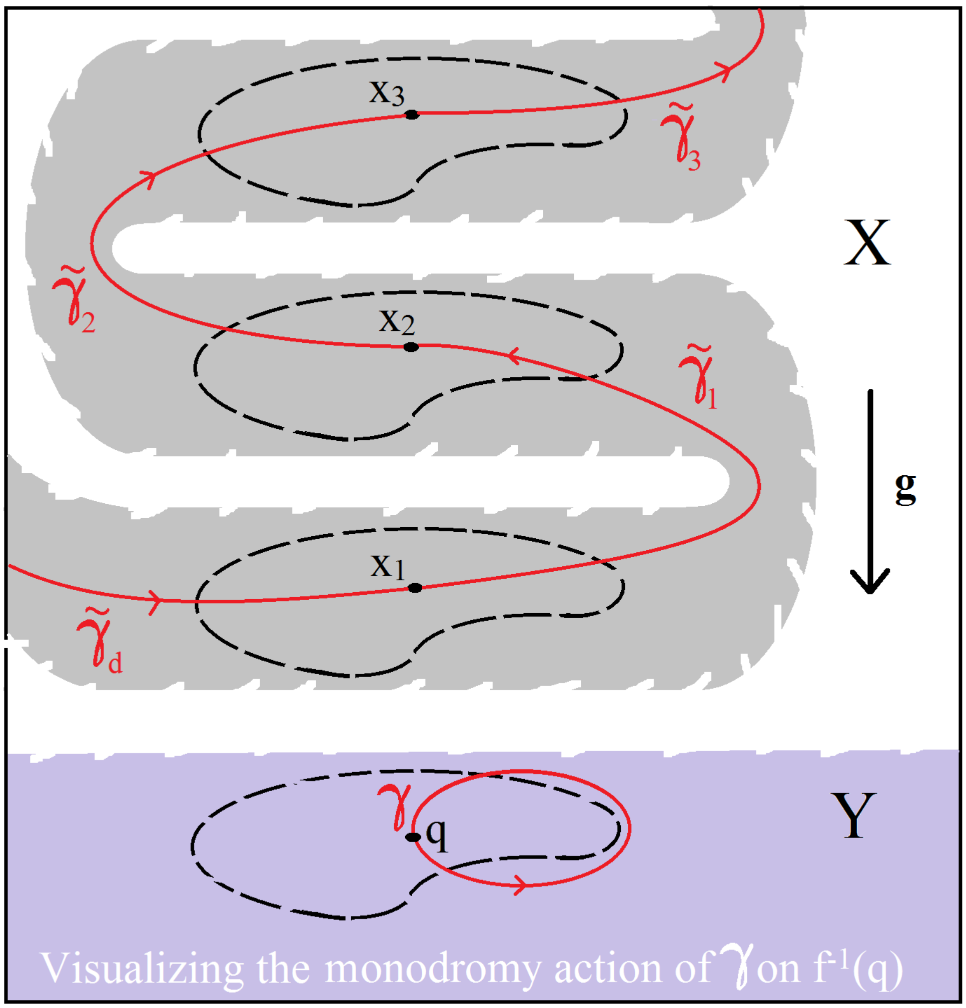
We are now ready for the final construction.

4: The Monodromy of (Finite Degree) Covering Spaces

Etymologically speaking, according to wikipedia:

> *"[M]onodromy is the study of how objects from mathematical analysis, algebraic topology, algebraic geometry and differential geometry behave as they "run round" a singularity. As the name implies, the fundamental meaning of monodromy comes from "running round singly". It is closely associated with covering maps and their degeneration into ramification"[5].*

As we saw in the previous section, the degree of the connected covering map corresponded to the index of the Galois-related subgroup. We take advantage of this and define a *new* construction below.



<< The Monodromy Representation Construction >>

Suppose for nice enough $\boldsymbol{Y}$ with connected covering space $\boldsymbol{g : X \to Y}$, that

$$\boldsymbol{d := deg(g) = [\pi_1(Y,q) : H]} \qquad \text{is finite.}$$

Then denote the fiber over the basepoint q by: $\boldsymbol{g^{-1}(q) = \{x_1, ..., x_d\}}$.

If we choose $[\gamma] \in \boldsymbol{\pi_1(Y,q)}$, we can lift γ uniquely to d different paths in the total space starting at each x_i. Label these $\boldsymbol{\{\widetilde{\gamma}_1, ..., \widetilde{\gamma}_d\}}$. The collection of end-points of each of these paths again forms the fiber $g^{-1}(q)$, which labelling correctly gives:

$$\boldsymbol{\{\widetilde{\gamma}_1(1), ..., \widetilde{\gamma}_d(1)\} = \{x_{\sigma(1)}, ..., x_{\sigma(d)}\}}$$

for some permutation $\boldsymbol{\sigma \in S_d}$ (the symmetric group).

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• Def: The **monodromy representation** of a (finite degree), connected covering space $g : X \to Y$, for Y “nice enough” and Galois-correspondent $H \leq \pi_1(Y, d)$, is just the group homomorphism:

$$\boldsymbol{\rho : H \to S_d}; \qquad \boldsymbol{\rho : [\gamma] \mapsto \sigma}$$

as we have defined above.

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Special note here, S_d can then be represented as permutation matrices of some vector space as we have seen in representation theory. The composition of the two homomorphisms gives us something more tractable to work with, for which we can apply theory of characters etc.

Closing remarks: *(Requires some background in Riemann Surfaces)*
As hinted at in the quote starting this section, on (pg.87) Miranda goes on to describe Monodromy representations as they apply to *holomorphic maps between Riemann surfaces*. If we take out the *branch points* from the image of the map and the whole fiber of each branch point from the domain (which includes *ramification points*, then each point in the base space has the same amount of pre-images (i.e. the degree of the map), so the holo becomes a covering space and we can apply what we learned here.

5: An Example

The following is *Example 4.5 (pg.87 [4])*, observing the unramified holomorphic map given by the n^{th} power map.

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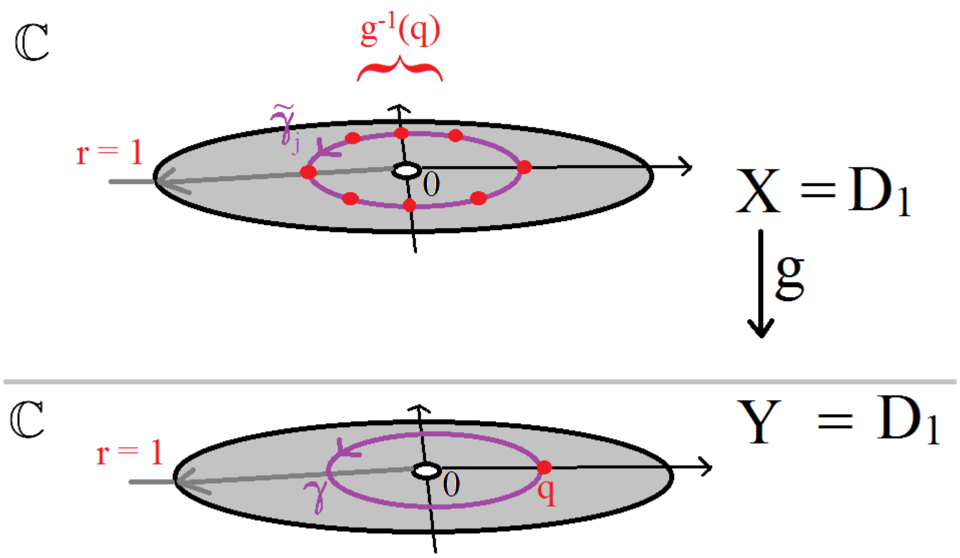

Take the map $\boldsymbol{g : X \to Y}$ of punctured discs as in the figure to be given by $\boldsymbol{y = g(x) = x^n}$. Since this is a map from $\mathbb{C}$ to $\mathbb{C}$, we can readily speak of it being holomorphic. And it is, since it's complex derivative is given by $y' = nx^{n-1}$.

We want to observe as well, that g is already in *local normal form* with respect to the two charts for X and Y centered at zero's, as g is its own *global coordinate representation*. By Lemma 4.4 (p.45 [4]), since $y' \neq 0$ for all points in X (recall 0 was not included), this implies all points have *multiplicity* 1 (so we really have an unramified map). Moreover, taking any nonzero point in the image and looking at the sum of the multiplicities of points in the fiber, we deduce that $deg(g) = n$.

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Let $\boldsymbol{q := \frac{1}{2^n}}$ be the base point in D_1. If we let $\boldsymbol{\zeta_n := e^{2\pi i/n}}$ be the *primitive* n^{th} *root of unity*, then:

$$\boldsymbol{g^{-1}(q) = \{\zeta_n^j/2 \mid j \in \{0, ..., n-1\}\} \equiv \{x_1, ..., x_n\}}.$$

The generator $[\gamma]$ for the fundamental group $\boldsymbol{\pi_1(Y, q)}$ is given by the loop:

$$\boldsymbol{\gamma(t) := \frac{1}{2^n} e^{2\pi i t} \text{ for } t \in [0, 1].}$$

This loop lifts to the *paths*:

$$\widetilde{\gamma}_j(t) = (\zeta_n^j/2)e^{\frac{2\pi i}{n}t} \text{ for } t \in [0,1],$$

($j \in \{0, ..., n-1\}$) whose starting and ending points are respectively at:

$$\widetilde{\gamma}_j(0) = \zeta_n^j/2 \qquad \text{and} \qquad \widetilde{\gamma}_j(1) = \zeta_n^{j+1}/2.$$

Therefore the **monodromy representation** is generated by the n-cycle. That is:

$$\rho : \pi_1(Y, q) \to S_n \text{ via}$$

$$[\gamma] \mapsto (12 \; ... \; n).$$

(Again, $[\gamma] := \frac{1}{2^n}e^{2\pi i t},\ t \in [0,1]$). We can of course further represent this as a permutation matrix on the space $\mathbb{C}^n$, with respect to the standard basis:

$$\left[\rho([\gamma])\right]_{\beta_{std}} = \begin{bmatrix} 0 & 0 & ... & 0 & 1 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

So $\chi_\rho([\gamma]) = 0$ and the rest of the characters are produced by powers of the matrix above. ■

Challenges:

1.) Find the character of the monodromy/permutation representation for the holomorphic map between complex tori:

$$\boldsymbol{F : X \to X; \quad x \mapsto 2x}$$

where $X := \mathbb{C}/L$ for some lattice L in the complex plane.
[Hint: This is an unramified map between (connected) Riemann surfaces.]

2.) Prove the bijective correspondence discussed in Section 3.3.[1] Then using the definition of *Galois connection* given in the references [3] show this bijective correspondence yields a Galois connection.

3.) Prove the Proposition at the end of Section 3.3.[1]

4.) Prove the existence of simply connected Universal Covers for "nice" spaces. [1]

5.) Find a monodromy/permutation representation for a "*branched covering*" given by a ramified holomorphic map (that is, remove the appropriate points from the domain and codomain and compute as we have before). Then see how the ramification induces a new *cycle structure* in matrix form (see what happens to the character of the representation). [Hint: See the discussion on pg.87-88 [4] together with our example above. Then notice Lemma 4.6 (p.88 as well).]

References

[1] A. Hatcher: Algebraic Topology, Cambridge University Press (2002).

[2] D. Dummit, R. Foote: Abstract Algebra, John Wiley and Sons (2004).

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[4] R. Miranda: Algebraic Curves and Riemann Surfaces, American Mathematical Society (1995).

[5] Wikipedia contributors. Monodromy. Wikipedia, The Free Encyclopedia. May 2, 2019, 17:48 UTC. Available at: https://en.wikipedia.org/w/index.php?title=Monodromy&oldid=895204765. Accessed April 19, 2020