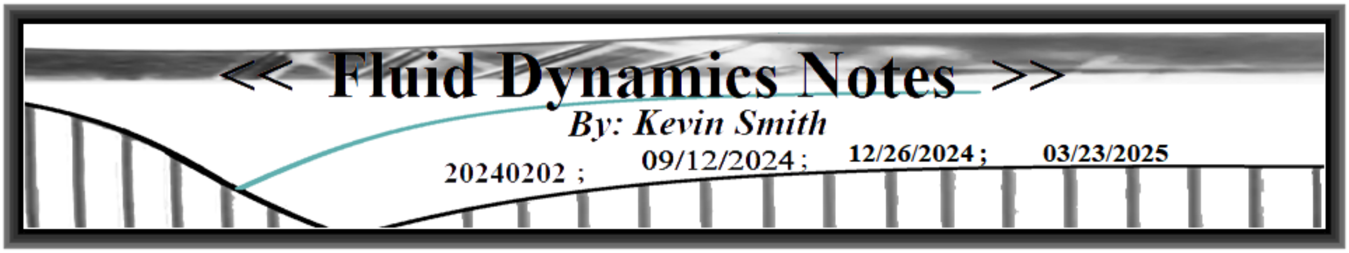
<< Fluid Dynamics Notes >>
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20240202 ; 09/12/2024 ; 12/26/2024 ; 03/23/2025

<< Chapter I - Introduction >>

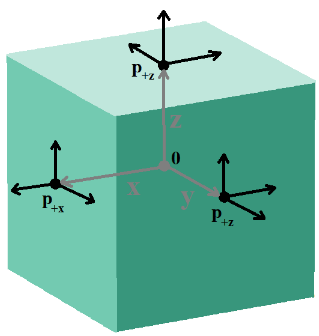
Sections 1.1-1.4 are mostly cited from Chapters 1 and 2 of [3], with more direct paraphrasing or quotes given page numbers. The rest is from my own scientific background.

1.1: Fluids and Forces

• Def: A **fluid** is a collection of particulate matter (atoms or molecules) whose *relative structure deforms continuously under the application of a continuous tangential force.*

Ex: Liquids, gases, or vapors are considered fluids. They may consist of multiple elements.

• Def: A **control volume** is a containing figure that we use to study a fluid by way of coordinates. The geometric boundary of which we call the **control surface**.



• Def: Designate the faces of a cube by their normal displacement from the center of mass as in the above figure. Then take $\boldsymbol{p_{\pm i}}$ to indicate the center point of the corresponding planes. We may write down the image of a **force field**, $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$, at these center points as:

$$\boldsymbol{F}(\boldsymbol{p}_{\pm i}) = \big(\boldsymbol{F}^x(\boldsymbol{p}_{\pm i}), \boldsymbol{F}^y(\boldsymbol{p}_{\pm i}), \boldsymbol{F}^z(\boldsymbol{p}_{\pm i})\big)$$

For the choices of planes: $\prod_{+x}$, $\prod_{+y}$, and $\prod_{+z}$, the collection:

$$\begin{bmatrix} \boldsymbol{F}(\boldsymbol{p}_{+x}) \\ \boldsymbol{F}(\boldsymbol{p}_{+y}) \\ \boldsymbol{F}(\boldsymbol{p}_{+z}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}^x(\boldsymbol{p}_{+x}) & \boldsymbol{F}^y(\boldsymbol{p}_{+x}) & \boldsymbol{F}^z(\boldsymbol{p}_{+x}) \\ \boldsymbol{F}^x(\boldsymbol{p}_{+y}) & \boldsymbol{F}^y(\boldsymbol{p}_{+y}) & \boldsymbol{F}^z(\boldsymbol{p}_{+y}) \\ \boldsymbol{F}^x(\boldsymbol{p}_{+z}) & \boldsymbol{F}^y(\boldsymbol{p}_{+z}) & \boldsymbol{F}^z(\boldsymbol{p}_{+z}) \end{bmatrix} =: \begin{bmatrix} \eta_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \eta_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \eta_{zz} \end{bmatrix}$$

is referred to as a **stress tensor**. Note that there are other choices possible for such a matrix, depending on which ordering you choose and which sign (i.e. it's not unique). Sticking with all positives, upgrading this to a *matrix field* would just require a *variable center of mass.*

The $\boldsymbol{\tau_{ij}}$ are called **shear stresses** and the $\boldsymbol{\eta_{ii}}$ **normal stresses**. In our definition of a fluid, connotate "shear stress" with "tangential force". One also sees the term **flux**, for forces that can permeate a surface. Roughly, this is just $\int_S \boldsymbol{F} \cdot \boldsymbol{n} d\boldsymbol{A}$.

• <u>Def:</u> A **fluid particle** or **fluid element** is an abstracted, averaged mass and volume entity for an atom or molecule in the fluid. Think of the cube above with a weight listing as under the elements in the *Periodic Table*.

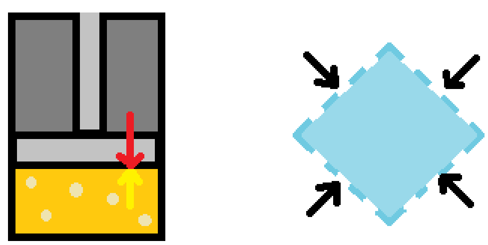
• <u>Def:</u> **Fluid density** about a point can be defined as the amount of mass per unit of volume centered at that point. Instead of specifying the particular geometry of the volume or the center point, we can say implicitly:

$$\boldsymbol{M}_{\text{in V}} = \int_{\boldsymbol{V}} \boldsymbol{\rho}(\boldsymbol{x}) \boldsymbol{dV}.$$

We will observe situations later, where time variance is included and this definition becomes useful.

• <u>Def:</u> We assume sufficient density of our fluids as to avoid considering *vacuum* effects. This is more or less the **fluid continuum** assumption (not to be confused with "continuum hypothesis"). In certain contexts (upper atmosphere or quantum level), this assumption is no longer valid.

• <u>Def:</u> **Compressibility** is the *ability (or lack thereof) to increase the fluid density*, $\boldsymbol{\partial_t \rho(x,t) > 0}$, at a given point by applying a converging force field or a force field against a solid boundary. If a fluid cannot have its density increased in this manner with a reasonably large magnitude force, it is said to be *incompressible* ($\boldsymbol{\partial_t \rho \equiv 0}$). *Incompressible fluids can be used to transmit forces between pistons, turbines, etc.*



• <u>Def:</u> Another related quantity to compression is in the word... **pressure (centered at** p**)** is the measure of force/flux exerted on a patch of surface (averaged by its surface area). For total **pressure** on an object, just extend over the whole surface. *This leads to a discussion on* ***buoyancy****, when compared to gravity.* In general:

$$\boldsymbol{Pressure} := \frac{\int_S (\boldsymbol{F} \cdot \boldsymbol{n}) \boldsymbol{dA}}{\int_S \boldsymbol{dA}}.$$

[*Project: (Phenomenology) There is a relation between pressure, density, temperature, heat transfer, condensation, evaporation, container volume, etc. (the state of a fluid) and the restriction/expansion of the volume with flow velocity change. Lookup "HVAC Systems" and "orifice tubes". Otherwise explore this and stay tuned for Chapter 3!*]

(Continues)

• <u>Def:</u> **Work** is a measure of force applied over a distance. One can imagine a piston in an engine acting on an air/fuel mixture to compress it. On the other hand, work is also done post ignition on the piston driving the crankshaft etc. $\boldsymbol{W = F \cdot d}$ or $\boldsymbol{W = \int F(x)dx}$

• <u>Def:</u> **Energy** is the capacity of an entity to do work. The previous example had energy stored in the air/fuel mixture, that through combustion performed work. Similarly, on the electrical side, lead-acid batteries have the potential to do work, built up from the chemical reaction in the form of free electrons. Energy comes in various forms and we will have more to say about this as time goes on.

But we are still in the introductory phase. So next, we list some features of fluids in motion...

1.2: Flow Geometry

• Def: Take a **fluid flow** to be a time-dependent mapping of space:

$$\boldsymbol{\theta} : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$$

$$(\boldsymbol{p}, \boldsymbol{t}) \mapsto \boldsymbol{\theta}(\boldsymbol{p}, \boldsymbol{t})$$

subject to:

$$\boldsymbol{\theta}(\boldsymbol{\theta}(\boldsymbol{p}, \boldsymbol{s}), \boldsymbol{t}) = \boldsymbol{\theta}(\boldsymbol{p}, \boldsymbol{s} + \boldsymbol{t}) \text{ and } \boldsymbol{\theta}(\boldsymbol{p}, 0) = \boldsymbol{p}$$

to describe the locations of centered fluid elements (pg.209 [6]). This is where the continuum assumption comes in, since we might otherwise trace vacuous points. The added conditions ensure that we can trace a point in an easy way (as opposed to recursion). This actually describes a *right*-group action so that we get *time-orbits* etc. It would also be convenient to assume $\boldsymbol{\theta} \in \boldsymbol{C}^0$ (at least) so that we get well definition of the following:

• Def: From continuous flows, we have associated **velocity fields** (a.k.a. **flow fields**),

$$\boldsymbol{X} : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$$

$$(\boldsymbol{p}, \boldsymbol{t}) \mapsto \boldsymbol{X}_{\boldsymbol{p}}(\boldsymbol{t}) := \partial_t \boldsymbol{\theta}(\boldsymbol{p}, \boldsymbol{t}).$$

If $\forall \boldsymbol{p} : \forall \boldsymbol{t} : \ \boldsymbol{X}_{\boldsymbol{p}}(\boldsymbol{t}) = \boldsymbol{X}_{\boldsymbol{p}}(0)$, we say the flow is **steady**. That is, the flow field at each point is constant in time. Consider straight line and circular steady flow for two examples.

(Back in [3]):
• Def: Some related curves of interest in the analysis are listed below:
> A **timeline** (through $\boldsymbol{p_0}$) is the collection of points initially vertical to $\boldsymbol{p_0}$ and normal to a shearing force. These lines track deformation.

> A **pathline** (through $\boldsymbol{p_0}$) is the temporal image of $\boldsymbol{p_0}$ in the flow:

$$\forall \boldsymbol{t} : \ \gamma_{path,p_0}(\boldsymbol{t}) := \boldsymbol{\theta}(\boldsymbol{p_0}, \boldsymbol{t}).$$

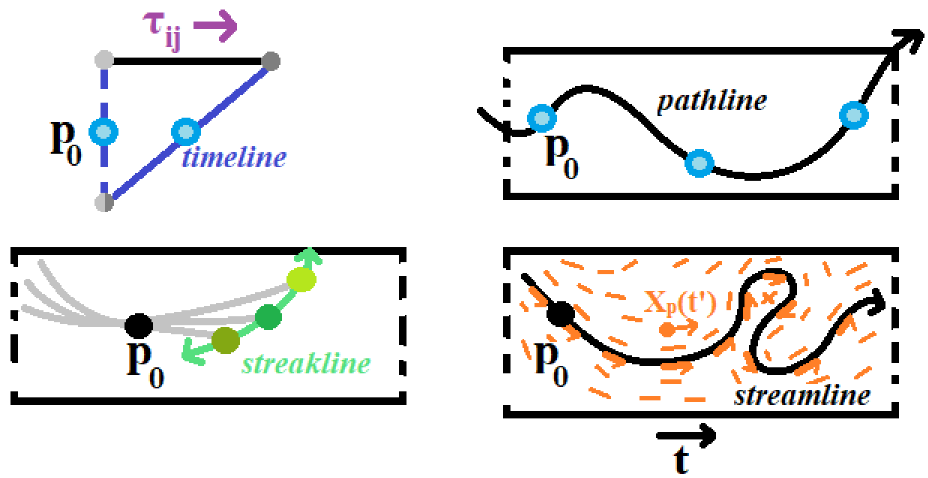
> A **streakline** (through p_0) is the curve through all fluid elements (at time t) that passed through the same spatial point (at some time).

$$\forall \boldsymbol{t} : \ \gamma_{streak,p_0}(\boldsymbol{t}) = \left\{ \boldsymbol{\theta}(\boldsymbol{p}, \boldsymbol{t}) \;\middle|\; \exists \boldsymbol{t_0} : \gamma_{path,p}(\boldsymbol{t_0}) = \boldsymbol{p_0} \right\}$$

> A **streamline** (through p_0) is like an *evolving integral curve* for the flow field. That is, at each point in time, the streamline is tangent to the flow. Its parameter-velocity vector at each point in space equals the flow fields time-velocity vector. In symbols:

$$\forall \boldsymbol{s} : \forall \boldsymbol{t} : \ \left[\partial_s \gamma_{stream,p_0}(\boldsymbol{s}, \boldsymbol{t}) = \partial_t \boldsymbol{\theta}(\gamma(\boldsymbol{s}, \boldsymbol{t}), \boldsymbol{t}) = \boldsymbol{X}_{\gamma(s,t)}(\boldsymbol{t}) \right].$$

Visuals for the above:



★ **Prop:** For *steady flows* pathlines, streaklines, and streamlines coincide (p.23).
[Exercise: Prove this!]

• Def: Note that contexts for flows are either **internal**, **external**, or **open-channel**. Respectively, the flow must be bounded by a solid surface (e.g. going through a pipe), effectively unbounded (as in airflow around a wing), or bounded but filled with multiple separated fluids.

• Def: One may describe flows as either **laminar** (with structure separating into layers) or **turbulent** (chaotic structure that doesn't separate nicely). This is seen below from [8]:

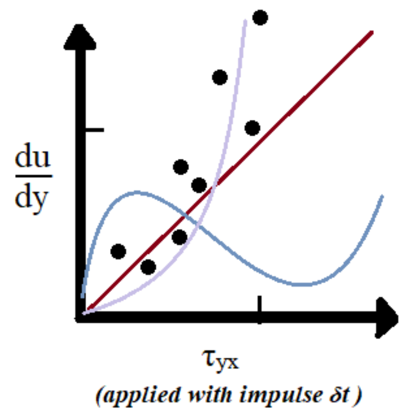
Here the timelines are becoming more chaotic.

Next, we describe an attribute that plays a role in this and other types of flow evolution...

1.3: Characterizing with Viscosity

The same applied force can affect two fluids in different ways. One fluid can also react to the same force differently when the application interval changes. This reaction can be inferred in each case through experimental data or just considered in the abstract. Either way, we would like to algebraically categorize fluids and this is a way to do so.

For example, consider the hypothetical data in the below graph:



We would like to say:

$$\frac{du}{dy} = f(\tau_{yx}) \approx c_2(\tau_{yx})^2 + c_1(\tau_{yx}) + c_0.$$

That is, the *deformation rate*, $\frac{du}{dy}$, is proportional to the square of the *shear stress*, τ_{yx}. Again this is a hypothetical situation. In the literature, one encounters the reverse dependency and sees mostly basic polynomials of the forms:

$$\tau_{yx} = \mu\frac{du}{dy} \quad \text{or} \quad \tau_{yx} = k\left(\frac{du}{dy}\right)^n$$

• Def: In the first case, the constant μ is called **absolute (dynamic) viscosity**. In the second case, the quantity $k\left|\frac{du}{dy}\right|^{n-1}$, is called **apparent viscosity**, k is the **consistency index**, and n is the **flow behavior index**. Another related quantity is the **kinematic viscosity**, which is $\nu = \frac{\mu}{\rho}$ (absolute viscosity over density at a point). As well, our model equations can have τ_{yx}-intercepts, "c_0", that are nonzero–this offset is called a **yield stress**.

(Continues)

• <u>Def:</u> Fluids satisfying the first equation are called **Newtonian fluids**. Otherwise, **Non-Newtonian**. "Fluids in which the apparent viscosity decreases with increasing deformation rate are called **pseudoplastic** (or shear thinning) fluids. If the apparent viscosity increases with increasing deformation rate, the fluid is termed **dilatant** (or shear thickening). Fluids that behave as solids until minimum yield stresses are exceeded and subsequently exhibit a linear relation betweeen stress and deformation rate are referred to as an **ideal** or **Bingham plastic**. Fluids that show a decrease in apparent viscosity with time under a constant applied shear stress are called **thixotropic**. Fluids that show increase in apparent viscosity with time are termed **rheopectic**. Some fluids after deformation, partially return to their original shape when applied stress is released; such fluids are called **viscoelastic**." (p.32-33)

Experimental results can be grouped together under these names and others of course. In the next section, we start the analysis...

1.4: Fluid-Solid Interaction

Thus far, our major assumptions have been:

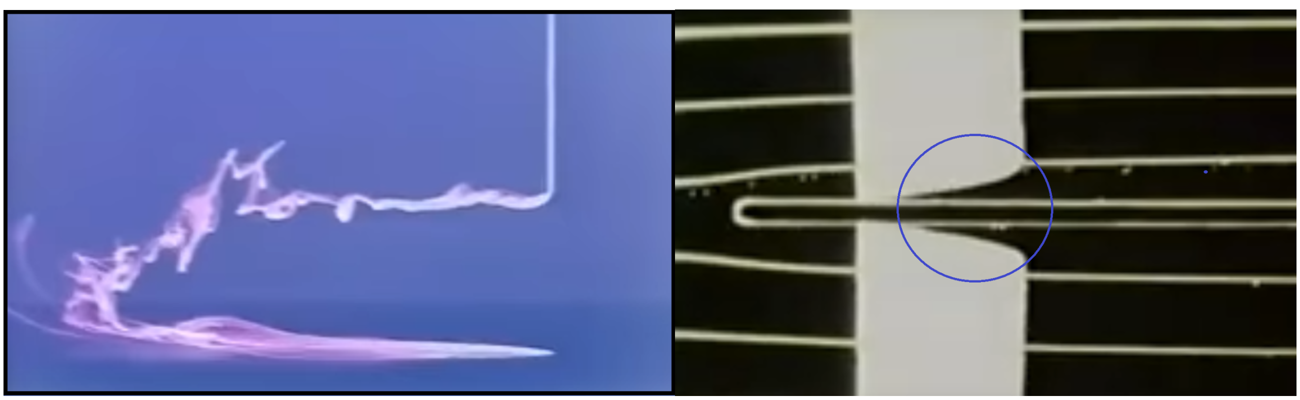
(i) Fluids deform continuously under continuous shear stress,

(ii) Fluids are either compressible or *effectively* not compressible, and

(iii) Fluid density is nowhere zero (and positive). (To avoid vaccuums etc.)

• Def: *(iv)* (p.3 [3]) (Phenomenology) The **no-slip condition** is the *experimental observation* that in a small neighborhood of the solid surface, the fluid "making contact" has zero relative velocity. This microscopic, space-filling phenomenon translates to a frictional force that propogates away from the surface.

• Def: (pg.35 [3]) The extent of the fluid experiencing this frictional force is called the **boundary layer**. Outside of this layer is the **inviscid layer**, where it flows without obstruction.

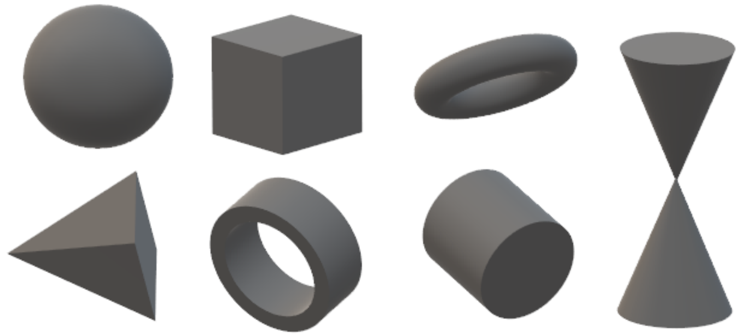


Experimental Images [15, 8]

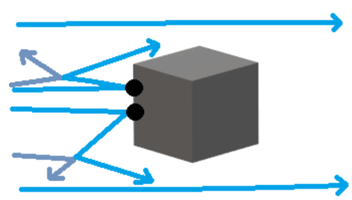
(Left) A die is injected at the solid surface and lifted away as time goes on. The flow is from right to left. Notice how the contact fluid seems to be trapped under the current.

(Right) We see flow from left to right over a flat plate. The dark lines are fluid and the light lines are air streaklines with one large timeline going vertically across. The boundary layer is developing as the dark two sided wedge in the middle.

A Thought Experiment: External Flows
Imagine other solids, suspended in a flowing fluid with gravity present:

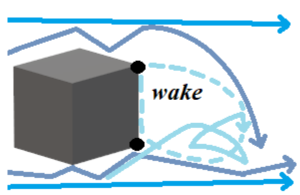


Taking the cube for example (leaving the rest for later), one might think that fluid particles would either pass over/under it unobstructed or crash into it and rebound in various ways, subsequently colliding with the fluid behind it and pushing it back into the cube, until it has found its way around. According to the literature (pg.36-37 [3]), points on the surface for which this happens are called **stagnation points**.



There should be a momentum/energy transfer from the fluid to the solid, but with subsequent collisions, this momentum may be regained for the primary fluid particles. The (time-dependent) net force experienced from such collisions is called **drag**. The region in front of the cube face no doubt experiences turbulence.

After the fluid falls over the rear end, it should collide with the layer below it leaving more turbulence until it later stabilizes. Directly behind the cube, there should be a region where only splashed particles enter. Let us call the trailing region a **wake**. Moreover, let us call the initial boundary of the wake, **separation points**.



There is an abundance of questions we can ask: What happens to the attributes of the fluid in the regions mentioned above (i.e. density, pressure, viscosity)? How does the boundary layer develop there with respect to the no-slip condition? How would compressability change the story? What about temperature, flow velocity, angle of the cube, whether or not it is mounted? How do we optimize drag and wake size by altering the geometry or otherwise? ...

See the **NCFMC** video on boundary layer control for more on this.[21]

(Continues)

<< Chapter II - On Analysis and Experiment >>

Sections 2.1-2.2 are based on multiple sources. Including Ch.4-6 [3], Ch.3 [7], and articles [13, 9, 10, 11, 16].
Section 2.2b also includes [6]. Section 2.3 is a bulk reference to the filmography [21]. And Section 2.4 is from Ch.7 in [3].

2.1: Physical System Laws

Next, we wish to develop some useful equations for predictive purposes and analysis.
We will be using the following well studied/verified (a priori inertial) equations:

- **Mass Law:**

$$\frac{d}{dt}M_{system} = 0$$

- **Momentum Laws (Linear & Angular):**

$$\frac{d}{dt}P_{system} = \sum_i F_i$$

$$\frac{d}{dt}H_{system} = \sum_i \tau_i$$

- **Energy Law:**

$$\frac{d}{dt}E_{system} = \frac{d}{dt}(Q + W)$$

- **Entropy Law:**

$$\frac{d}{dt}S_{system} \geq \frac{1}{T} \cdot \frac{d}{dt}Q$$

• Def: To elaborate: M stands for the total *mass*; P is total *linear momentum*, $\boldsymbol{p} = m\boldsymbol{v}$ at the particle level; H is total *angular momentum*, $\boldsymbol{h} = m\boldsymbol{\omega}$ at the particle level; F_i's are force fields; τ_i's are sources of *torque* exertion (other fields basically with nonzero *curl*); E stands for total *energy*, Q total *heat* in the system, W is total *work* done on the system (force applied over a distance), S is total *entropy* or disorder measure, T is temperature of the system and Q again is *heat*. Care is to be taken as to the direction of application of work and heat flow for the signs.

These laws are known respectively as "**Conservation of Mass**", "**Newton's Second Law**", "**Moment of Momentum**", and the "**First-**" and "**Second Law(s) of Thermodynamics**".

We need to phrase these in terms of **control volumes** in order to meet applications...

2.2: Control Volume (Intensive) Formulation

- Def: Each of the 5 equations have their own **system (a.k.a. extensive) properties**:

$$\boldsymbol{M}, \boldsymbol{P}, \boldsymbol{H}, \boldsymbol{E}, \text{ and } \boldsymbol{S}$$

that we witness the time variation of and relate to the RHS's via their corresponding laws. We can relate each of these to a **density (a.k.a. intensive) property**, or property per infinitesimal volume (or per infinitesimal mass if we include mass density, ρ) by way of integration:

$$\boldsymbol{M}_{\text{(total in V)}} = \int_V (\rho 1) dV$$

$$\boldsymbol{P}_{\text{(total in V)}} = \int_V (\rho v) dV$$

$$\boldsymbol{H}_{\text{(total in V)}} = \int_V (\rho \omega) dV$$

$$\boldsymbol{E}_{\text{(total in V)}} = \int_V (\rho e) dV$$

$$\boldsymbol{S}_{\text{(total in V)}} = \int_V (\rho s) dV$$

Think dimensionally. For example $\frac{\text{mass}}{\text{volume}} \frac{\text{energy}}{\text{mass}} = \frac{\text{energy}}{\text{volume}}$. Notice too that P and H are vector quantities, so we are compactly stating multiple component equations. These imply:

- **Mass Law (2):**

$$\frac{d}{dt}\left[\int_V \rho(x,t) dV\right] = 0$$

- **Momentum Laws (Linear & Angular) (2):**

$$\frac{d}{dt}\left[\int_V \rho(x,t) v(x,t) dV\right] = \sum_i F_i(x,t)$$

$$\frac{d}{dt}\left[\int_V \rho(x,t) \omega(x,t) dV\right] = \sum_i \tau_i(x,t)$$

- **Energy Law (2):**

$$\frac{d}{dt}\left[\int_V \rho(x,t) e(x,t) dV\right] = \frac{d}{dt}(Q+W)(x,t)$$

- **Entropy Law (2):**

$$\frac{d}{dt}\left[\int_V \rho(x,t) s(x,t) dV\right] \geq \frac{1}{T} \cdot \frac{d}{dt} Q(x,t)$$

2.2b: Further Evaluations *

The form of equation we wish to consider from the previous section is:

$$\frac{d}{dt}\int_V f(x,t)dV \sim g(x,t).$$

Based on considerations which will become clear after the fact, we continue as follows...

A Moving Volume of Fluid in its Flow Field:

Suppose M is a **compact**, **orientable** k**-dimensional manifold with boundary** hypersurface, ∂M. Let this represent a collection of fluid particles.

Then suppose we have a **continous family of orientation-preserving diffeomorphisms** (smooth maps with smooth inverses whose differential doesn't mix the coordinate bases), that represent *displacement* and *deformation* of the manifold over time:

$$\{F_t : M \xrightarrow{o.p.diff.} M_t\}_{t\geq 0}$$

We have by Prop 16.6(d) (p.407 [6]), a.k.a. **"Diffeomorphism Invariance of the Integral"** that for each F_t and **differential form** $\omega \in \Omega^k(T^*M_t)$, the integrals are related via the **pullback**:

$$\int_M F_t^*(\omega) = \int_{M_t} \omega$$

It should be noted that the variable of integration in each respective case is $p \in M$ and $q \in M_t$. But we may also assume ω is time-dependent, varying independently from the manifold. That is:

$$\int_M (F_t^*(\omega))(p,t) = \int_{M_t} \omega(q,t)$$

Differential forms restricted above a **chart** and expanded in a **local frame** look like:

$$\omega\big|_{F_t(U_i)} = \omega_I dy^I = \omega_{i_1\ldots i_k} dy^{i_1} \wedge \ldots \wedge dy^{i_k}$$

here the **summation convention** is applied and the **multi-indices** are of course *strictly increasing*.

In general the pullback is defined by pushing forward vector fields and applying the form there, but in the special case of **top-degree forms** (i.e. **volume forms**, where $k = dim(M)$) we have the **coordinate expression for the pullback** in compatible charts (Prop 14.20 p.361 [6]) [augmented with time-variance]:

$$(F_t^*(\omega)) = (\omega_{1\ldots k} \circ (F_t, Id))(det(dF_t))dx^1 \wedge \ldots \wedge dx^k$$

Applying this, we have:

$$\int_{F_t(U_i)} \omega = \int_{U_i} (\omega_{1\ldots k} \circ (F_t, Id))(det(dF_t))dx^1 \wedge \ldots \wedge dx^k$$

|<<|

More cleanly, if $\omega = f_{i,t}(q,t)dV_{i,t}$ is the representation above a deformed chart, this becomes:

$$\int_{F_t(U_i)} f_{i,t}(q,t)dV_{i,t} = \int_{U_i} (f_{i,t} \circ (F_t, Id))(det(dF_t))dV_{i,0}(p,t)$$

We are now ready to look at the derivative of the intensive integral (over a chart) from the previous section. By definition with the difference quotient, we have:

$$\frac{d}{dt}\bigg|_{t_0} \int_{U_i(t)} f_{i,t}(q,t)dV_{i,t} = \lim_{t\to 0} \frac{1}{t}\left[\int_{F_{t_0+t}(U_i)} f_{i,t_0+t}(r, t_0+t)dV_{i,t_0+t} - \int_{F_{t_0}(U_i)} f_{i,t_0}(s,t_0)dV_{i,t_0}\right].$$

Applying the diffeomorphism invariance of the integral and the pullback formula:

$$= \lim_{t\to 0} \frac{1}{t}\left[\int_{U_i} (f_{i,t_0+t} \circ (F_{t_0+t}, Id))(det(dF_{t_0+t}))dV_{i,0} - \int_{U_i} (f_{i,t_0} \circ (F_{t_0}, Id))(det(dF_{t_0}))dV_{i,0}\right].$$

Applying linearity of integrals, now that the domain is time-independent, we can pass the limit and fraction inside, and factor out the differential:

$$= \int_{U_i} \lim_{t\to 0} \frac{1}{t}\Big[(f_{i,t_0+t} \circ (F_{t_0+t}, Id))(det(dF_{t_0+t})) - (f_{i,t_0} \circ (F_{t_0}, Id))(det(dF_{t_0}))\Big]dV_{i,0}$$

Applying the definition of derivative again, we have:

$$= \int_{U_i} \frac{\partial}{\partial t}\bigg|_{t_0} \big[(f_{i,t} \circ (F_t, Id))(det(dF_t))\big](p,t)dV_{i,0}$$

Since the chart was arbitrary, let us suppress some notation with the *assumption* that a **partition of unity** makes this local expression global:

$$\frac{d}{dt}\int_{M_t} f(q,t)dV_t = \int_{M_0} \frac{\partial}{\partial t}\big[f(F_t(p),t) \cdot det(dF_t(p))\big]dV_0$$

Applying the Product Rule, Linearity, and then the Chain Rule (to the first term), we can open this up more. Next page:

$$= \int_{M_0} \left[\frac{\partial}{\partial t}(f(F_t(p),t)) \cdot det(dF_t(p)) \right] dV_0 + \int_{M_0} \left[f(F_t(p),t) \cdot \frac{\partial}{\partial t}(det(dF_t(p))) \right] dV_0$$

$$= \int_{M_0} \left[\sum_{j=1}^{k} \frac{\partial f(F_t(p),t)}{\partial \#_j} \cdot \frac{\partial F_t^j(p)}{\partial t} + \frac{\partial f(F_t(p),t)}{\partial t} \right] \cdot det(dF_t(p)) dV_0$$

$$+ \int_{M_0} \left[f(F_t(p),t) \cdot \frac{\partial}{\partial t}(det(dF_t(p))) \right] dV_0$$

We can now pullback to M_t via the inverse of F_t, splitting the terms via linearity again:

$$= \int_{M_t} \left[\sum_{j=1}^{k} \frac{\partial f(F_t(p),t)}{\partial \#_j} \cdot \frac{\partial F_t^j(p)}{\partial t} \cdot det(dF_t(p)) \right] \circ (F_t^{-1}, Id) \cdot det(dF_t^{-1}) dV_t(q,t)$$

$$+ \int_{M_t} \left[\frac{\partial f(F_t(p),t)}{\partial t} \cdot det(dF_t(p)) \right] \circ (F_t^{-1}, Id) \cdot det(dF_t^{-1}) dV_t(q,t)$$

$$+ \int_{M_t} \left[f(F_t(p),t) \cdot \frac{\partial}{\partial t}(det(dF_t(p))) \right] \circ (F_t^{-1}, Id) \cdot det(dF_t^{-1}) dV_t(q,t).$$

Simplifying a bit, we get:

$$= \int_{M_t} \left[\sum_{j=1}^{k} \frac{\partial f(q,t)}{\partial \#_j} \cdot \frac{\partial q^j}{\partial t} \cdot det(dF_t(F_t^{-1}(q))) \right] \cdot det(dF_t^{-1}(q)) dV_t$$

$$+ \int_{M_t} \left[\frac{\partial f(q,t)}{\partial t} \cdot det(dF_t(F_t^{-1}(q))) \right] \cdot det(dF_t^{-1}(q)) dV_t$$

$$+ \int_{M_t} \left[f(q,t) \cdot \frac{\partial}{\partial t}(det(dF_t(F_t^{-1}(q)))) \right] \cdot det(dF_t^{-1}(q)) dV_t.$$

Then since determinant is multiplicative, $det(dF_t(F_t^{-1}(q))) \cdot det(dF_t^{-1}(q)) = det(Id_{TM_t}(q)) = 1$ implies:

$$= \int_{M_t} \left[\sum_{j=1}^{k} \frac{\partial f(q,t)}{\partial \#_j} \cdot \frac{\partial q^j}{\partial t} \right] dV_t$$

$$+ \int_{M_t} \left[\frac{\partial f(q,t)}{\partial t} \right] dV_t$$

$$+ \int_{M_t} \left[f(q,t) \cdot \frac{\partial}{\partial t} \Big(ln\big(det(dF_t(F_t^{-1}(q)))\big) \Big) \right] dV_t,$$

or more concisely, summarizing:

$$\frac{d}{dt} \int_{M_t} f dV_t = \int_{M_t} \langle \nabla f, v \rangle dV_t + \int_{M_t} \partial_t f dV_t + \int_{M_t} f \partial_t \Big(ln\big(det(dF_t|_{F_t^{-1}})\big) \Big) dV_t,$$

where $v := \partial_t q$ and ∇ is the spatial gradient. (Continues)

Applying the following identity to the first term on the RHS,

$$\langle \nabla f, v\rangle = \langle \nabla, fv\rangle - f\langle \nabla, v\rangle$$

followed by Linearity, definition of divergence and the Divergence Theorem gives:

$$LHS = \int_{\partial M_t} \langle fv, n\rangle dA_t - \int_{M_t} f \cdot div(v) dV_t + \int_{M_t} \partial_t f dV_t + \int_{M_t} f\partial_t \Big(ln\big(det(dF_t|_{F_t^{-1}})\big)\Big) dV_t,$$

Lastly, as a matter of preference, rearrange and apply Linearity once more:

$$\frac{d}{dt}\int_{M_t} f dV_t = \int_{\partial M_t} \langle fv, n\rangle dA_t + \int_{M_t} \partial_t f dV_t + \int_{M_t} f \Big[\partial_t \Big(ln\big(det(dF_t|_{F_t^{-1}})\big)\Big) - div(v)\Big] dV_t, \quad (\bigstar)$$

where again $v(q,t) := \partial_t q = \partial_t F_t(p)$, ∇ is the spatial gradient, and $n(q,t)$ is an outward-unit normal field to the boundary. p is a point on the initial manifold and $q = F_t(p)$ is its image point on the deformed manifold. So $v(p,t)$ is the velocity of the particle $p = F_t^{-1}(q)$ at time t in the flow field. *Think of a bubble flowing down stream from an initial point, this equation describes a difference between internal properties at those times when the ghosts are infinitesimally separated.*
If the velocity field of the ***current density****, say* $fv_{current}$*, is different than* $v = \partial_t F_t(p)$*, we can find a section of the time-varying vector bundle (since fibers are forgetfully abelian groups),* $v_{relative\ displacement}$*, such that*

$$v_{current} - v_{\text{displacement}} = v$$

Interpretation and Reductions:

We have the flux of the **current density**, fv, through the boundary plus time variance of the **density**, f, and some **mystery quantity** in the volume's interior, all contributing to the time variance of the **extensive property**. *And this is based solely off of analysis in the differential geometry.*

We used a family of orientation-preserving diffeomorphisms (continuous in the time index), $\{F_t\}_{t\geq 0}$, to both be convenient with pullbacks of forms and integrals AND to reconcile with our previous assumption (i) (see Section 1.4), that fluids deform continuously.

If we are to have ($\bigstar$) stand up to the other assumptions (ii-iv) as well as previous work with flows and flow fields (in Section 1.3), what do we get?

Well, (ii) says fluids are either compressible or not; (iii) says our fluids have everywhere strictly-positive density, and (iv) is the no-slip condition (regarding interaction with solid boundaries –not the manifold's).

We have (iii) implies $f(x,t) > 0$ identically. No reductions here. (iv) helps to define $v(q,t)$, but yields no reductions. (ii) says either the density remains constant in time, $\partial_t f(q,t) \equiv 0$ OR NOT. *This yields a reduction in the second term in the incompressible case!*

(Continues)

As far as the flow and flow field notation goes, $F_t(p) \equiv \theta(p,t)$ and hence:

$$\forall p : \forall t : \; X_p(t) = \partial_t \theta(p,t) = \partial_t F_t(p) = v(p,t).$$

Lastly, we encountered the notion of *steadiness*. This now says:

$$\forall p : \forall t : v(p,t) = v(p,0) \quad \text{or} \quad \partial_t(\partial_t F_t(p)) \equiv 0.$$

Integrating this last statement for steadiness twice over time says:

$$F_t(p) = C_1(p)t + C_2(p)$$
$$\implies dF_t(p) = dC_1(p)t + dC_2(p)\ .$$

$$\implies \partial_t(ln(det(dF_t(p)))) = \partial_t\Big(ln\big(det\big(dC_1(p)t + dC_2(p)\big)\big)\Big)$$
$$= \frac{1}{det\big(dC_1(p)t + dC_2(p)\big)} \cdot \partial_t det\big(dC_1(p)t + dC_2(p)\big)$$
$$= \frac{det(dC_1(p))}{det\big(dC_1(p)t + dC_2(p)\big)}.$$

Which can clean up the expression at least if we can model the flow with explicit $C_j(p)$'s. As well,

$$\implies div(v(p,t)) = \sum_{i=1}^{k} \partial_i \partial_t F_t^i(p) = \sum_{i=1}^{k} \partial_i C_1^i(p).$$

[Exercise: Calculate these terms for $C_1(p) := (p^1, ..., p^k)$ and $C_2(p) \equiv 0$ and come up with other instances for the $C_j(p)$'s.]

Note that for $C_1(p) \equiv 0$ and $det(dC_2(p)) \not\equiv 0$, we get the whole mystery term vanishes!

This analysis yielded reductions of (★) for the case of special $F_t(p)$ (in steady flow) and for incompressible fluids. To reiterate, these results are an expansion of the introduced Laws!

2.2b': The Differential Formulation for Conservation Laws *

The equation from the last subsection:

$$\frac{d}{dt}\int_{M_t} f dV_t = \int_{\partial M_t} \langle fv, n\rangle dA_t + \int_{M_t} \partial_t f dV_t + \int_{M_t} f\left[\partial_t\Big(ln\big(det(dF_t|_{F_t^{-1}})\big)\Big) - div(v)\right] dV_t$$

can have a different form. If we didn't apply Divergence Theorem to the first term and combine:

$$\frac{d}{dt}\int_{M_t} f dV_t = \int_{M_t}\left[div(fv) + \partial_t f + f\left[\partial_t\Big(ln\big(det(dF_t|_{F_t^{-1}})\big)\Big) - div(v)\right]\right] dV_t$$

For Conservation Laws, the LHS vanishes. So if:

$$\partial_t f = -div(fv) + f\left[div(v) - \partial_t\Big(ln\big(det(dF_t|_{F_t^{-1}})\big)\Big)\right]$$

holds, the extensive property is "conserved".

/<</

(Continues)

2.3: Library of Measuring Devices

From the collection of videos from NCFMC, suggested by the main reference, I've collected the following list and re-organized them.

	⟨⟨APPARATUS NAME ⟩⟩	⟨⟨UTILITY ⟩⟩
1	Multimedia Chamber	General
2	Fan/Propellor	Stream Induction
3	Stream Variance Device	Shear Stress Induction
4	Source/Sink	Vorticity Induction
5	Centrifuge	Rotational Shear Induction
6	Diffuser/Variable Geometry Chamber	Pressure/Speed Variation, Flow Separation, etc.
7	Scale and Fulcrum	Drag Measurement
8	Gravity Tubes	Terminal Velocity, Drag, etc.
9	Oscillators	Wave Induction
10	High Speed Camera	Slow Motion Visualization
11	Strobe Lighting	Slow Motion Visualization
12	Bubble Jet and Colored Die	Visualization of Streamlines etc.
13	Manometer	Pressure/Speed Visualization
14	Vorticity Meter	Visualization of Vorticity
15	Viscometer	Visualization of Viscosity
	...	...

We have the ability to open or close a system of fluid and/or air to observe flow properties; To aid in observation, we set up cameras and indicators that mark desired points; We can induce motion (linear, rotational, wave, etc.); And we can alter the shape of the container and affect the state of the fluid with more clever devices.

Knowledge of existing phenomena warrants architecture of experiment. Ignorance of phenomena requires *experimentation* in the other sense of the word. Extreme or combinatorial/categorical situations may exhume facets we might otherwise overlook.

(>> Fwd to Appendix 1).

2.4: Obtaining Data Points with Cost Efficiency

We wish to model **Fluid-Mechanical Properties**, p_{m+1}, with experimentally derived, measurable **(suspected parameter)** values: $p_1, ..., p_m$. That is,

$$p_{m+1} = p_{m+1}(p_1, ..., p_m).$$

The cost of obtaining this curve depends on that of measuring each plot point in a grid-range. Reduction in quantity, m, of parameters reduces total cost. The following is to that affect.

• Def: Allow us to define abstractly, **(base) parameters/dimensions**: $\{\beta_1, ..., \beta_n\}$.

Next, define **(regular) parameters**: $\{p_1, .., p_m\}$, such that:

$$\forall i : \exists \ (a_{i1}, ..., a_{in}) \in \mathbb{Z}^n : \ \left[p_i := \beta_1^{a_{i1}} \cdot \ ... \ \cdot \beta_n^{a_{in}}\right].$$

Then, define **(pre-dimensionless) parameters**: $\{X_1, ..., X_k\}$, where:

$$\forall j : \exists (b_{j1}, ..., b_{jm}) \in \mathbb{Z}^m : \ \left[X_j := p_1^{b_{j1}} \cdot \ ... \ \cdot p_m^{b_{jm}}\right].$$

The expansion of X_j in terms of β_k's through p_i's looks like:

$$X_j = \left(\beta_1^{a_{11}} \cdot \ ... \ \cdot \beta_n^{a_{1n}}\right)^{b_{j1}} \cdot \ ... \ \cdot \left(\beta_1^{a_{m1}} \cdot \ ... \ \cdot \beta_n^{a_{mn}}\right)^{b_{jm}}$$

Applying *Rules of Exponents* and *Commutativity of Multiplication-of-Dimensions*, we obtain:

$$X_j = \beta_1^{(a_{11}b_{j1}+a_{21}b_{j2}+...+a_{m1}b_{jm})} \cdot \ ... \ \cdot \beta_n^{(a_{1n}b_{j1}+a_{2n}b_{j2}+...+a_{mn}b_{jm})}.$$

• Def: From here, we define **(dimensionless) parameters**, as those X_j such that:

$$Ab_j := \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{j1} \\ \vdots \\ b_{jm} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} =: 0.$$

Restating:

$$X_{j,\text{dimensionless}} := p_1^{b_{j1}} \cdot ... \cdot p_m^{b_{jm}} \leftrightarrow (b_{j1}, ..., b_{jm}) \in ker(A) \cap \mathbb{Z}^m.$$

How you calculate these ***integer-vectors*** in practice is on you. Typically, row reduction would be expected to work, but deeper Linear Algebra may need be applied in general.

There is a distinct method called the **Buckingham Pi Theorem**, that also has a Generalization available. [Exercise: Run through examples in Ch.7 [3]].

Note that $dim\left(ker(A) \cap \mathbb{Z}^m\right) \leq m$. And further, **rational independence** considerations may reduce further. That is, ensuring $X_j \neq X_j(\{X_i\}_{i \neq j})$.

2.4b: Geometric, Kinematic, Dynamic, and Incomplete Similarity

It turns out, that the set of reduced dimensionless parameters, $\{X_1, ..., X_r\}$, have significance other than reduction in cost. They can help with experimental results transference...

• Def: A **prototype** here means a full-scale experimental apparatus. A **model** here means a **scaled**, $(x : y) \in \mathbb{Q}$, version of the prototype.

• Def: If $\{X_j\}_{j,\text{prototype}} = \{X_j\}_{j,\text{model}}$, we can talk about **complete dynamic similarity** of results obtained in corresponding experiments.

As per (p.305+ [3]), we have **geometric similarity** when "*the model and prototype have the same shape, and that all [geometric] dimensions of the model be related to corresponding dimensions of the prototype by a constant scale factor*".

Two flows are **kinematically similar** if "":

$$\hat{v}_{prototype} = \hat{v}_{model} \quad \text{and} \quad |v_{prototype}| = \lambda |v_{model}|,$$

that is unit-directions of velocity fields are the same and magnitudes related by a scale factor.

Stability-regime similarity occurs when the PVT diagrams are the same or when both exhibit the same turbulence, cavitation, stall, stability, etc. [See [21] videos for more on this].

When applied forces present in both are such that "":

$$\hat{F}_{prototype} = \hat{F}_{model} \quad \text{and} \quad |F_{prototype}| = \lambda |F_{model}|,$$

we have **dynamic similarity**. **(Complete) fluid-dynamic similarity** occurs when similar relations hold for all relative parameters (viscosity, pressure, surface tension, etc), with the same scale factors.

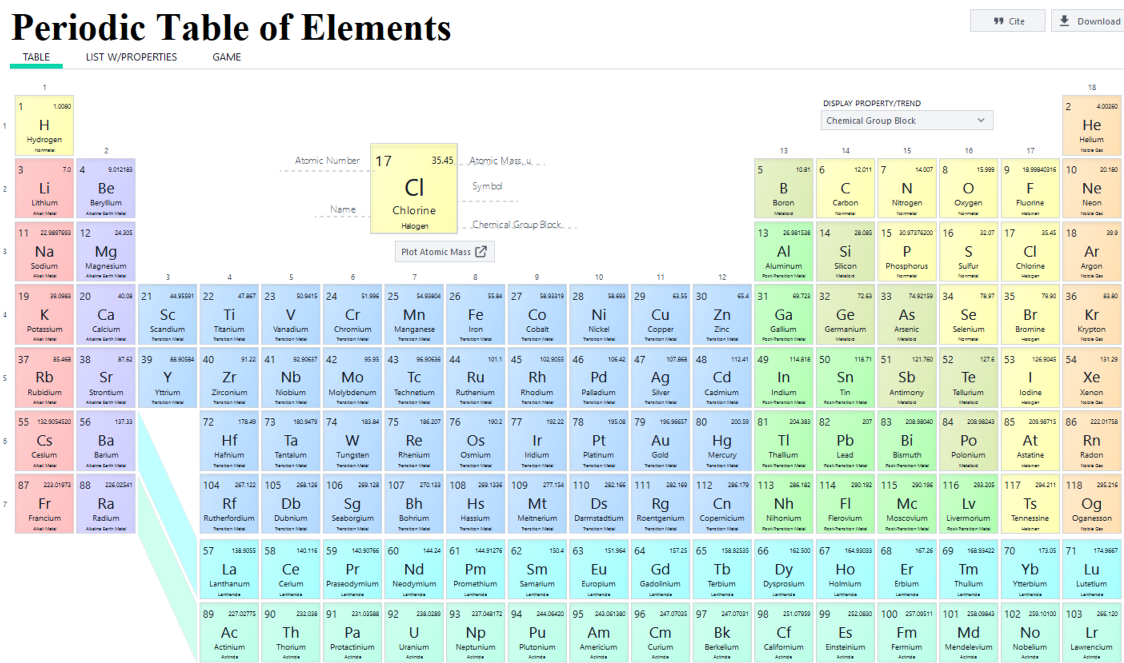
(Incomplete) similarity occurs when the X_j cannot be arranged to be all the same for reasons that become clear upon evaluation with the p_i's. In this event, further simulation such as triggering turbulence, must be done to achieve stability-regime etc.

• Def: Alternatively, (p.319 [3]) mentions flow **similitude (dynamic similarity)** occurring when flows have the same governing differential equations and boundary conditions (in dim'less form).

With esteem, special significance is given to dimensionless parameters found by: **Euler**, **Froude**, **Mach**, **Reynolds**, **Strouhal**, and **Weber**.

(Continues)

Section 1 references [19, 20, 18].



Let's start with a reference to a seemingly complete database on Earthly and Lab elements.

Aside from the main display of all elements, we have access to a(n):
- Plot of Atomic Mass vs. Atomic Number,
- Displays: Chemical Group Block, Atomic Mass (u), Standard State,
  Electron Configuration, Oxidation States, Electronegativity (Pauling Scale),
  Atomic Radius (Van der Waals) (pm), Ionization Energy (eV), electron affinity (eV),
  Melting Point (K), Boiling Point (K), Density (g/cm^3), Year Discovered.
- If you click an Element on the main display, it prints the Object Parameters for the element.
  Focuses on 1 and shows all displays in one box PLUS a link to a page on the element with more:
    Such as: Atomic Spectra, Ground Level, Estimated Crustal and Oceanic Abundance,
    History, Physical Description, Uses, Half-Life, Decay, Isotopes,
    Elemental Forms, Compounds, Handling and Storage, etc.

(Continues)

3.1: Properties of Mixtures

Elements can be combined into various mixtures, the properties of which can be tabulated/graphed for given experiments. Which properties we care about depends on the application.

A very superficial research elucidates properties as:

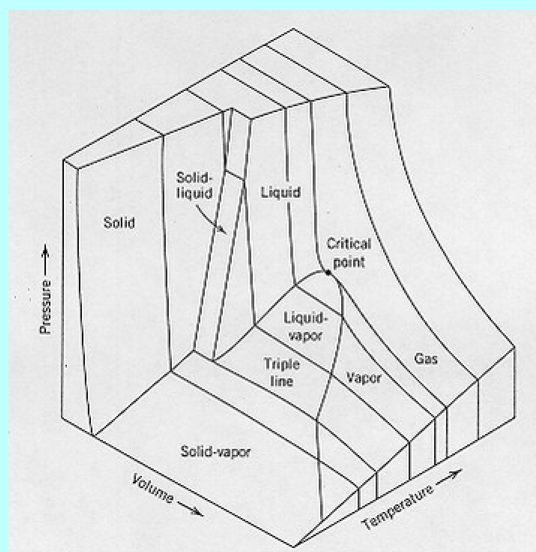
> **Chemical Composition**,
> **Boiling Point**,
> **Freezing Point**,
> **Latent Heat of Vaporization**
> **Thermal Conductivity**,
> **Specific Heat**, **Specific Volume**, **Specific Pressure**,
> **Corrosivity**, **Toxicity**, **Flammability**, **Explosivity**
> **Dissolvability**, **Di-electric strength**, and **Surface Tension**.

An example of a graph displaying important information is shown next:
• Def: The **PVT-Surface** of a(n) **element/compound/solution** is a graphical representation of the data triplet for **pressure**, **volume**, and **temperature** recorded in a lab.

PVT Surface

The (pvT) surface shown opposite is for a material that contracts on freezing. The surface is described by the Equation of State of the material and is the locus of all equilibrium states of the material.

The diagram shows regions associated with solid, liquid and gas phases. Regions in which two phases co-exist are normal to the (p,T) plane as in such regions isotherms and isobars are identical. As a phase change takes place, material goes from one phase to the other and the volume of the system changes. The Gibbs function is the same for the two phases in equilibrium.



(Continues)

An example of a more in depth data collection is found in the references [18] for **refrigerants** and is worth looking up.

This document lists Pressure vs. Enthalpy graphs with what seem to be lines at equilibrium such as **isotherms**, etc.

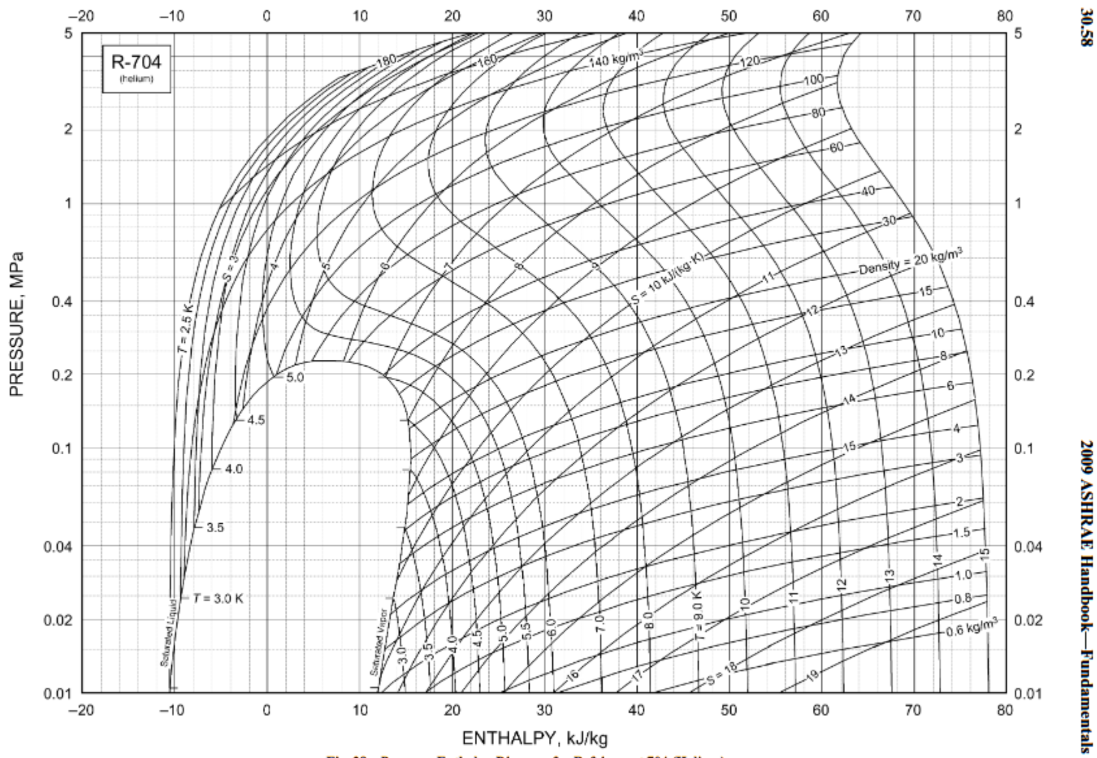


Fig. 28 Pressure-Enthalpy Diagram for Refrigerant 704 (Helium)
Note: The reference states for enthalpy and entropy differ from those in the table.

Contained also within the document are some more key terms, of which are new to us here:

> **Thermodynamic** and **Transport Properties**,
> **Temperature Glide**,
> **Critical Temperature**,
> **Enthalpy**,
> **Entropy**,
> **Bubble Line**, **Dew Line**, and
> **Equation of State**.

I'll leave proper discussion/elaboration of these terms to another author, awareness is the reachable goal here. What I would like to see next, to further the discussion of *analytical* properties of mixtures, is: What do **chemical equations** and **equations of state** look like? And to close the section, a seed for the Dynamic Effect of Molecular Combination on Phase Diagrams. Category of Phase Diagrams? Lol.

$$\boldsymbol{f(P, V, T) = PV - nRT = 0}$$

$$\boldsymbol{f(X_1, ..., X_r) = 0}.$$

See also JWL Equations; You've seen Stoichiometry right?

(Continues)

<< Chapter IV - Advanced Flow Geometry >>

This chapter has references [21] and prior knowledge.

Returning to the forte. In Chapter 1, we formalized flows and flow fields,

$$\boldsymbol{(p, t) \mapsto \theta(p, t)} \quad \text{and} \quad \boldsymbol{X_p(t) := \partial_t \theta(p, t)}.$$

We described special types of lines in flows that aid analysis with visualization. We set contexts, and later described viscosity, compressibility, and other related phenomena of fluids, such as boundary layers and turbulence. Chapter 2 went on to describe, roughly speaking, the effect of forces on a closed system (with expanded description in terms of current densities, $\boldsymbol{\rho(q, t) \cdot X_q(t)}$).

Here, we wish to return to our preliminary description of $\boldsymbol{(\theta, X)}$ (see Section 1.2), to get after *rotational motion*, *stratification*, *shock waves*, and perhaps better describe the *flow regimes* hinted at in the Appendix. The culminating topic will be the infamous *N-S equations*.

4.1: Rotational Motion

There are many options for describing rotations. To name a few, one can use:

> Roots of Unity, $\boldsymbol{\zeta_n^k}$, (for discrete rotations of 2D space witnessed as $\mathbb{C}$),
> Orthogonal Trig Matrices, $\boldsymbol{[c, s; -s, c](\theta)}$, (for continuous rotations of 2D space, $\mathbb{R}^{\boldsymbol{2}}$),
> Orthogonal Matrices in general (for analogous operations in $\mathbb{R}^{\boldsymbol{n}}$),
> Centripetal motion formula, $\boldsymbol{F = mv^2/r}$ (for kinematics of a single body),
> Cross product for torque, $\boldsymbol{\tau = r \times F}$ (for lever-arm-force application),
> Curl of Vector Fields, $\boldsymbol{curl(X) = \nabla \times X}$,
> Spinors, Quaternians, etc.

For continuous flows of a continuum of points (included into $\mathbb{R}^{\boldsymbol{3}}$), the best option to work with is the one for *curl of vector fields*. For each initial point, $\boldsymbol{p}$, flowed for a time, $\boldsymbol{t}$, we get a vector, $\boldsymbol{v := X_p(t)}$, at the point $\boldsymbol{\theta(p, t)}$.

The vector, $\boldsymbol{v}$, is the destination for the right hand rule, where $\boldsymbol{\nabla := (\partial_x, \partial_y, \partial_z)}$ is the starting point (think $\boldsymbol{(1, 1, 1)}$ at the original reference frame). When we curl the fingers from $\boldsymbol{\nabla}$ to $\boldsymbol{v}$, the thumb gives the direction of the *fixed vector* (the one normal to the plane of rotation). The *magnitude* of this fixed vector is the magnitude of the rotation at the point.

(Continues)

So, provided we have the mathematical description of a flow or flow field in *flat space*, we just need to analyze the following:

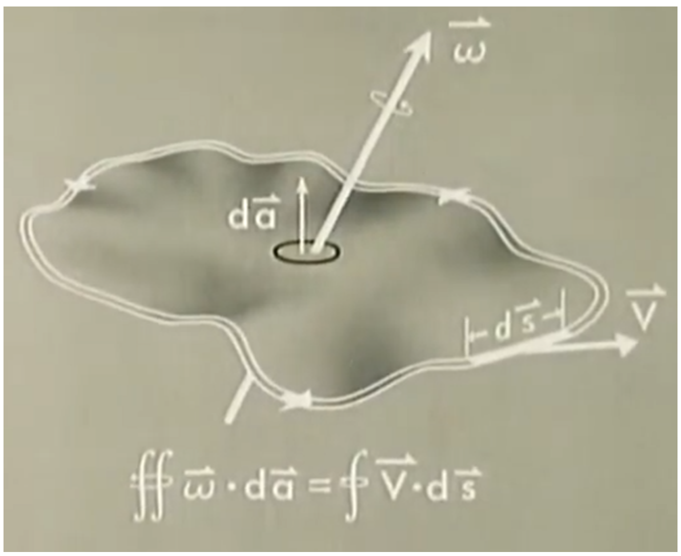
$$|\nabla \times X_p(t)| = |\nabla \times (\partial_t \theta(p,t))|$$

to see the rotation magnitude experienced by a fluid particle p at each instant. If we have this value identically zero, the point does not undergo rotations.

I mention "flat space", because across curved-manifolds, the field, ∇, may have a different expression at each point. This can be further convoluted with allowing for time-dependent curvy manifolds. Think $\boldsymbol{\nabla(\theta(p,t),t)}$.

Suppose we do not have a nice expression for $\boldsymbol{\theta(p,t)}$. As long as the flow is steady, we can obtain approximations experimentally/numerically from our visual aids. Even easier is to use a surface **vorticity meter** (little floaty golf ball with octants painted on it that we can see rotating as it travels downstream). More advanced sensors placed in an initial grid can surely be used to the direct effect.

We've described the tendency to rotate at each point in the field. What about in the phenomenonology? We have examples of: a draining volume (kitchen sink, bathtub, etc.); the trailing vortex of an airfoil; a tornado; etc.? The description relies on our knowledge of the flow field. But there is another notion we can take advantage of coming from Stokes' Theorem.



The figure above is from the corresponding video in [21]. This $\boldsymbol{\omega}$ vector represents our cross product when considering the bounding curve to be infinitesimal. As the curve expands its radius, the totality of such infinitesimal magnitudes is summed. Look at the double integral, this is the flux of $\boldsymbol{\omega}$ as a field! The closed curve, line integral on the right can be referred to as the **circulation**.

This value, with choice of macroscopic area, may be easier to experimentally acquire (say with varying size vorticity meters).

4.2: Refinement and Stratified Flow

The last section, from a conceptual standpoint, more or less took place on the surface of a flow. The developed expression for curl didn't. But this suggests the idea of: *below surface, or cross-sectional circulations being different than surface level.*

Stepping back from necessarily rotational motion, one can ask about **cross-sectional development of flows** and how differences might manifest? As the section title suggests, this is based off of analysis seen in [21] for Secondary and Stratified Flows.

Construction of Flow Expressions

• **Taylor/Laurent Series:**
For well-behaved, *steady* (linearly time-evolving), 1D flows of pure math form, one can consider polynomial expansions:

$$\theta(p,t) := \left(\sum_{i=a}^{b} c_i(p)^i\right)t,$$

where $(a,b) \in \{(0,n),(0,\infty),(-\infty,\infty),...\}$ is a discrete interval. The coefficients are related with **Taylor's Theorem**, **Cauchy's Integral Formula**, etc. via derivatives.

For higher dimensions, $p = (p^1,...,p^n)$, we have the total derivative, $df := \partial_j f \partial^j$, to work with and powers of these traces appear. [Exercise: Look up "Taylor Series in Two Dimensions".]

For *unsteady* (nonlinearly time-evolving), 1D flows, we have a simple example:

$$\theta(p,t) := \left(\sum_{i=a}^{b} c_i(p)^i\right)t^2.$$

Notice that the above covers transcendental functions such as: sin, cos, tangent, etc.

• **Newton's Law:**
When the flow is defined by a set of **super-posed** fields, we can derive the expression from Newton's Law.

• **Flow Combinations**
When we have two arbitrary flows, physically intersecting them doesn't just super-pose the flow fields (initially). Though long term, this may be the stabilization of the combined flow.

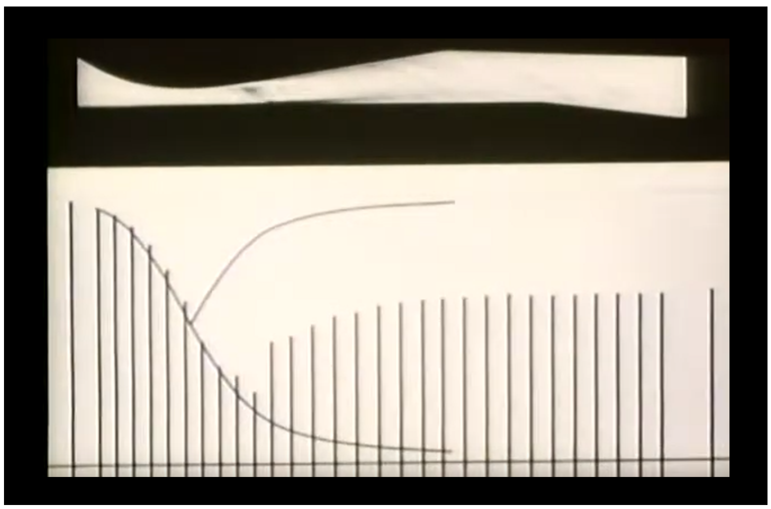
• **Limiting Flows**
In the same line of reasoning, we can take a sequence of flows and ask what the limiting flow is. This is an interesting caveat because: there are examples of sequences of differentiable curves whose limit is not differentiable. Among other Real Analysis type considerations. But, with foresight to shock-waves, this could be a useful analytical tool.

Now, we return to the main point... (Next page)

In the beginning of this section, we asked with impetus from the videos, how cross-sectional differences might manifest? In the above, subsection we actually got two answers. The first is, that *truncations* of our approximating series give different orders of accuracy to the flow. The first order approximation would then be the **primary flow**, the second order, the **secondary flow** and so on. Let's call this **refinement**.

As opposed to the physical combination of flows, where due to *chemical and buoyancy* considerations, a combination might separate and differ in bulk motion (e.g. oil and water), this is **stratification**.

4.3: Shock Waves and Regularity



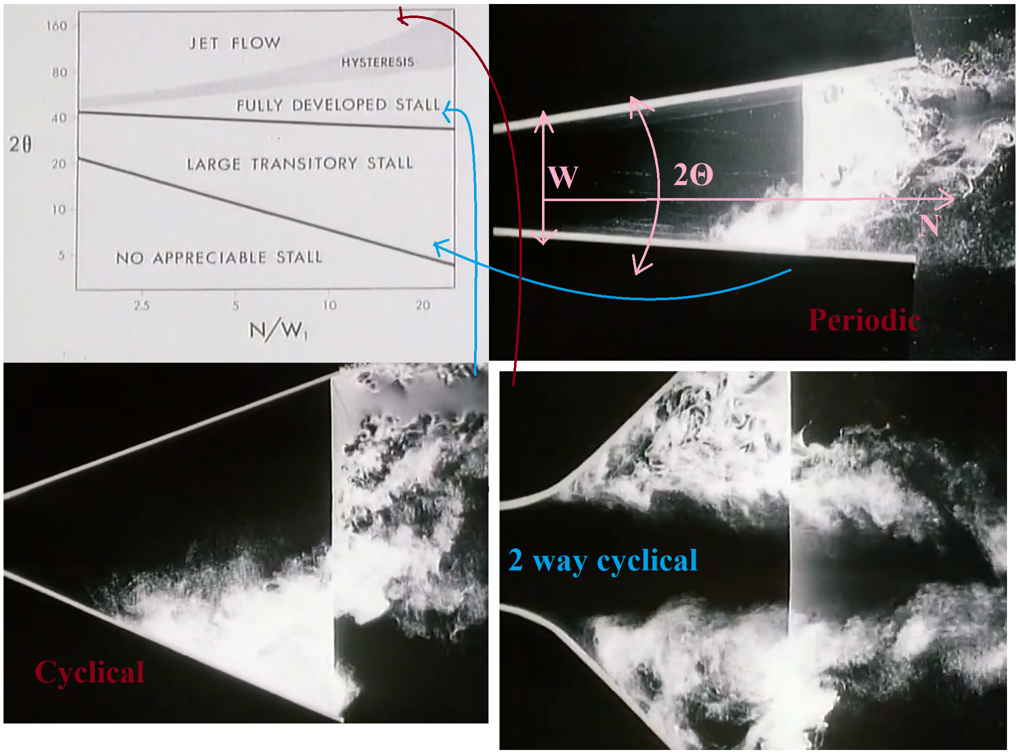
Shown above is the abstraction from the video in NCFMC for the *restricted* "Channel Flow of Compressible Fluid". Here we study the effects of the geometry of the wind tunnel in conjunction with increasing Mach number (i.e. increasing velocity). Initially, the manometer pressure curve is a bump with min at the choke point. As velocity increases, the smooth minimum point on the curve turns to a cusp. After a certain threshold velocity (Mach 1), the distribution is no longer symmetric (as can be seen above).

Further along in the video, multiple, variable geometry choke points are observed with new phenomena of "channel blocking" and "channel starting".

We can use our Physical System Laws (Section 2.1) to develop expressions for the flow velocity fields, particularly to relate pressure, area, density, and velocity. Because of the discontinuity in the slope of the pressure distribution, one might suspect discontinuities in the velocity field's as well.

This allows us to finish the section with an acquired concept of the study of $(\boldsymbol{\theta}, \boldsymbol{X})$ in different regularity classes, $\boldsymbol{C^0}, \boldsymbol{C^k}, \boldsymbol{C^\infty}, \boldsymbol{C^\omega}$, etc. Though according to the main text (p.647+ [3]), this discontinuity is only apparent.

4.4: Flow Regimes



The above collage was assembled from a video in NCFMC. These show continuous electrolysis timelines evolving with respect to diffuser output angle changes. The fact that there is backwards flow is called **stall**. Various levels of stall are witnessed with varying the ratio between length and width of the diffuser with the angle. Different regions of the $N/W - 2\theta$ plane are marked with names describing the character of the flow a.k.a. the **flow regime** (laminar, transitionary, turbulent, etc.). This resembles our phase diagrams (PVT surfaces).

Here, depending on viscosity and variables introduced by the diffuser, the boundary layer(s) change and affect the primary flow. Vortices propogate upstream and change the separation point. To analyze where the vortices begin and end amounts to analyzing the *zero sets of the magnitude of the curl of the flow field.* That is, where:

$$|\boldsymbol{curl(X_p(t))}| \approx \mathbf{0}.$$

The boundary layers may evolve and dissipate, only to recover stall states and dissipate periodically. Or they can be stable and not dissipate, in the case of *fully developed stall.* [Exercise: Research Hysteris and Jet Flow].

Turning over to turbulent flows. We would expect curl to be nonzero everywhere in the region. But, as stated in the "Turbulence" episode, there exist **energy cascades**. Energy dissipates symmetrically at given scales and has self-similarity as the scale decreases in the sense that it is (isotropic). Detecting this in the graph of $|\boldsymbol{curl(X_p(t))}|$ is a problem within itself.

Let us move on to the final section of this chapter, which is interested in the solutions for flow velocity, $\boldsymbol{u(x,t) := X_p(t)}$, in a given scenario.

4.5: Navier-Stokes

To directly quote from the millenium prize problem prompt [22] [with differential geometry notational changes: $\frac{\partial}{\partial x^i} = \partial_i$ and vector component indices raised]:

"The Euler and Navier Stokes equations describe the motion of a fluid in $\mathbb{R}^n$ ($n = 2$ or 3). These equations are to be solved for an unknown velocity vector $u(x,t) = (u^i(x,t))_{1 \le i \le n} \in \mathbb{R}^n$ and pressure $p(x,t) \in \mathbb{R}$, defined for position $x \in \mathbb{R}^n$ and time $t \ge 0$. We restrict attention here to incompressible fluids filling all of $\mathbb{R}^n$.
The *Navier Stokes* equations are then given by

$$(1) \quad \partial_t u^i + \sum_{j=1}^{n} u^j \partial_j u^i = \nu \Delta u^i - \partial_i p + f^i(x,t) \quad (x \in \mathbb{R}^n, t \ge 0),$$

$$(2) \quad \mathrm{div}(u) = \sum_{i=1}^{n} \partial_i u^i = 0 \quad (x \in \mathbb{R}^n, t \ge 0)$$

with initial conditions

$$(3) \quad u(x,0) = u^0(x) \quad (x \in \mathbb{R}^n).$$

Here $u^0(x)$ is a given, C^∞ divergence-free vector field on $\mathbb{R}^n$, $f^i(x,t)$ are the components of a given, externally applied force (e.g. gravity), ν is a positive coefficient (the viscosity), and $\Delta = \sum_{i=1}^{n} \partial_{ii}^2$ is the Laplacian in the space variables."

"Equation(s) **(1)** is just Newton's law $f = ma$ for a fluid element subject to the external force $f = (f^i(x,t))_{i \le i \le n}$ and to the **forces arising from pressure and friction**.

"Equation (2) just says that the fluid is incompressible."

To phrase in the terms of the "Fluid Dynamics of Drag Part III" video, newtons law says:

net pressure force + net viscous force = inertial force.

Above, we are looking at the material derivative on the LHS (inertial force) and on the RHS we have respectively: the viscous force (including ν), the pressure force (including p), and an externally applied force, f.

This reaches the scope of the advanced flow geometry I wanted to cover. Let's return for the finale on some practical applications...

(Continues)

<< Chapter V - Component and System Design >>

This chapter references a collection of wikipedia articles [23] as well as Ch.1 [5], Ch.2 and 3 [3], and some miscellaneous un-cite-ables.

With automotive application as the theme for this text, we will be building up to being able to study an *automatic transmission* (input/intermediate/output shafts, torque converters, gearsets (esp. planetary), clutches, hydraulic passages and electric solenoids, oil pumps, sensors, control modules, filters, housings, pressure-relief devices, etc.).

The purpose of a transmission is to transmit energy from the engine to the wheels in specific settings [P-R-N-(D)-L]. This of course has to further transmit through shafts, transfer cases, differentials, etc. The concept of **energy transfer** is a key player.

Starting from the point of view of early **waterwheels** (following wiki narrative), we will briefly consider particular types of transfers and move on to study **turbines** affected via jets etc., saving the tie to torque conversion with **fluid couplings** until the end.

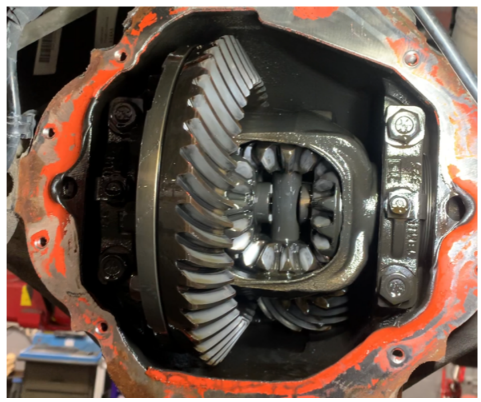
Following this will be a discussion of hydraulics (force/energy transfers via intermediate incompressible fluids). Tying this to valve bodies, it is only natural to go back and describe from there, the operation of the transmission.

(Continues)

5.1: Energy Transfer

• Def: As we've likely mentioned before, **Energy** is the capacity to do **Work**. It comes in various forms: **Potential**, **Kinetic**; **Electrical**, **Heat**, **Chemical**, etc. Some are arguably derived forms with Potential and Kinetic being primitive. Work done on a system gives it energy. Work done by a system reduces its energy.

• Def: (p.28-36 [5]) The transfer of energy can be considered on the macroscopic and microscopic scales. Especially in the context of fluids or electrical, **conduction** is the physical contact version at the particle level. Think *elastic/inelastic* collisions. But connotation is on the macro scale, whereby a string of particles stays put and assembly lines the energy from one end to the other. For examples: battery cables or the physical application of gear teeth to eachother, etc.



This is opposed to **convection**, where bulk relocation of containing particles, the energy is moved.

Similarly, **radiation** is like divergent convection, but more so via constituent particles, light waves, etc. This last one does not apply to us (hopefully).

(Continues)

5.2: Turbines

The above pictures were taken on the job. One of the techs took a blower motor and turned it into a load testing device similar to a light for trailer wiring (left). The (right) is the inlet for a turbo on a diesel engine, with the tube taken off between the air box. In reading the main text, [3] and the articles in [23], I started seeing these devices more.

Even when actuated, the pathlines, etc. are not visible, so it is hard to say what the device is doing - if it is being acted on or otherwise. We must use knowledge of the system and reasoning to say for sure. But, in the abstract, we can study both cases.

Contexts and Classifications:

Combinatorially thinking, these devices fit into the contexts we've previously discussed (**internal**, **external**, or **open-channel**) and they exhibit fluid-solid interactions. The fluid can be of varying **viscosity**, **flow regime**, **relative depth**, etc.; And the **material strength** (including thermo considerations) and **geometry** can be altered in the device. Let's reiterate the perspective of **work** as well.

In what follows, let us assume interaction with a fluid of fixed character, in the steady, laminar, flow regime. The material of the device is of fixed character as well. We allow **compressibility** to be included in the character. So that the only things changing in our problems are the:

Unordered Variables:

0.) **Flow Context**,
1.) **Relative Depth / Method of Interaction**,
2.) **Fluid Input State**, and
3.) **Blade Geometry/Device Orientation**, where
4.) **Work Direction** is either *"Exo"* or *"Endo"*.

Note that this is not necessarily an exclusive list. But rather something we can hope to apply our Physical System and Similitude Laws to shortly.

To cover some terms seen in the literature, let us do an aside before continuing.

• Def: In the above photos, we can see the blades of the devices are oriented such that the path of flow through them is perpendicular (resp. parallel) to the axis of rotation of the device. We may refer to these as **radial/centrifugal** vs. **axial** devices. [23].

• Def: In the perpendicular case (left photo), we can have input streams engaging the concave sides of the blades resulting in clockwise rotation, or convex to counter-clockwise. The position of the engagement has obvious terms like **overshot**, **undershot**, **backshot**, **breastshot**.

In the case of waterwheels, there is another relevant term that is the height of the original fluid source prior to flowing downhill and engaging the device (this is called the **head**) [23].

• Def: In the parallel case, we have additional terms as: **impulse** and **reaction turbines**, which refer to whether the stream hits at a point and deflects (as in the case provided by a *stationary* blade assembly (**stator**) acting like a nozzle unto a rotating blade assembly **rotor**);

Or, reaction if the stream rides along the blade and has pressure differentials affecting the rotor. Some designs "[vary] the degree of reaction or impulse from the blade root to its periphery".

In addition to such variance, we can also have **multi-stage** devices (think turbines in series).[23]

• Def: If we define **power** as the rate of change of energy (measured in Joules/sec = Watts), we can compare output vs input in a ratio called **mechanical efficiency**. Another force to torque input/output ratio is called **mechanical advantage**. [23]

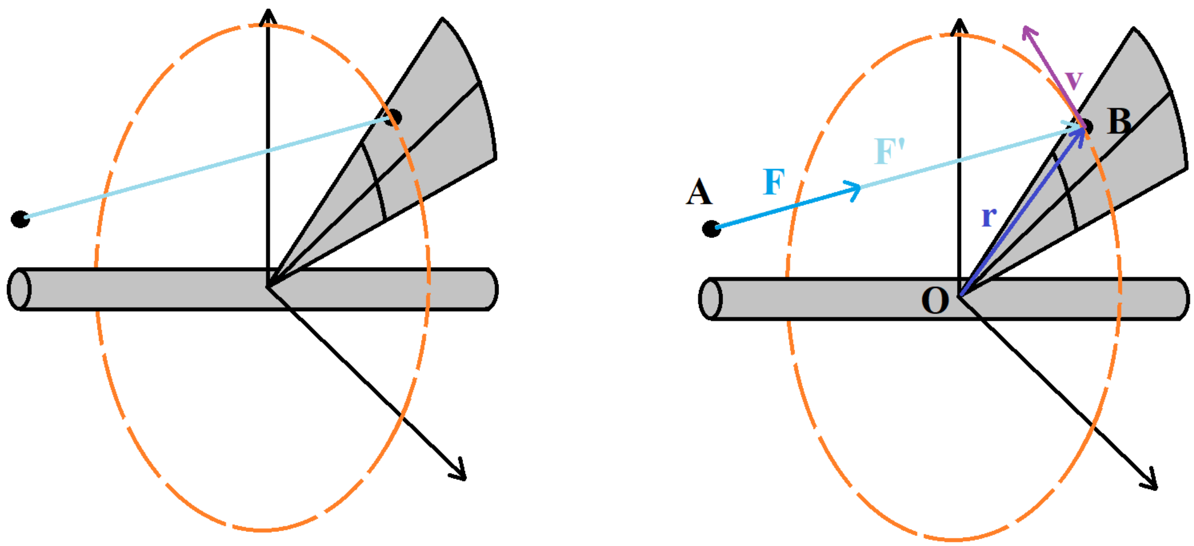
• Def: **Specific speed** is a number that "describes the speed of the turbine at its maximum efficiency with respect to the power and flow rate". [23]

Lastly, a seemingly fun fact that I found: "The number of blades in the rotor and the number of vanes in the stator are often two different prime numbers in order to reduce the harmonics and maximize the blade-passing frequency."

(Continues)

End aside. With (0-4) in mind, let's design a model to analyze.

Flow Context := External (open)
Flow Orientation := Axial
Method of Interaction := Single nozzle offset from centerline (breast shot).
Fluid Input State := Free/Sufficient Velocity
Geometry := Free/To Be Determined
Work Direction := Endo (Energy absorbed by the device).



We are dealing with force applied to a lever arm to generate torque on the output shaft. Simply put, our governing equation is:

$$\text{“}\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{comp_v(F)}\text{”}$$

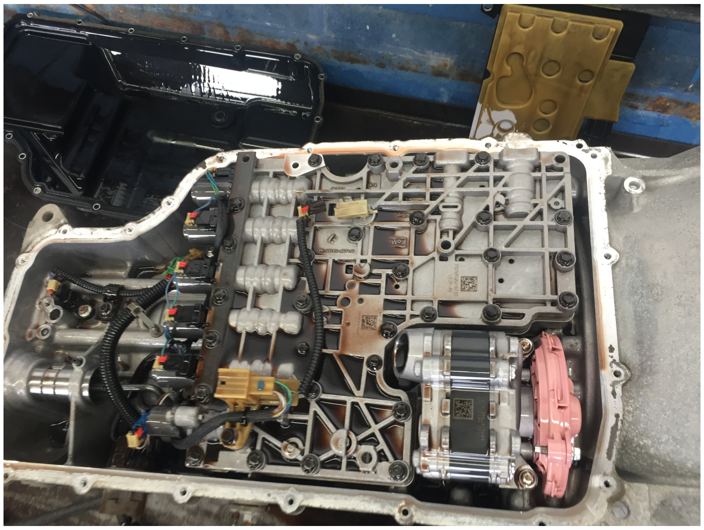
We wish to find the geometry of the blade and the positions of $\boldsymbol{A}$ and $\boldsymbol{B}$ such that the component of $\boldsymbol{F}$ in the direction of $\boldsymbol{v}$ is maximized. The fact that the fluid is impacting the blade means the process is endo and it is easy to see the rest of the hypotheses/definitions are satisfied in the picture. [Exercise: Continue on with this problem.]

The above neglected gravity and any other forces/torques we might consider. Our physical system law for angular momentum can be asserted here as well for the purpose of solving for other quantities (this is done in the main text, see (pg.145+ [3])).

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}\boldsymbol{H_{system}} = \sum_{\boldsymbol{i}} \boldsymbol{\tau_i}$$

and of course expanded with the intensive formulation to get some wild expression. Allegedly, this is where the Euler equation comes from. Rather than write all this out, let's move on...

5.3: Hydraulic Circuits



This is a picture of a "valve body" or "main control" taken from previous work on Ford's. One could also describe it as a ***hydraulic pressure control circuit*** *containing electrically actuated solenoids (coil/valves), commanded by the PCM through the connectors and pressurized by the oil pump (bottom right).*

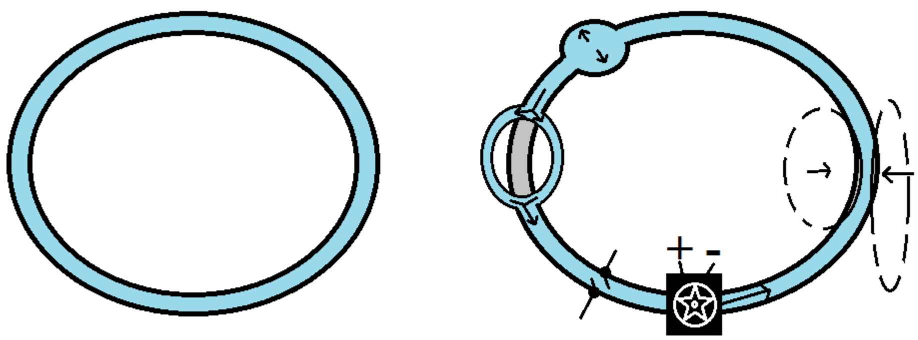
These paths, when open/pressurized affect clutches and hence gear applications. This statement doesn't hint at how complicated that gear application process is and how many components it takes (also including snap rings, planetary gear sets, O-rings, internal fluid passages, roller bearings?, etc.).

Recently, I discovered **hydraulic circuit diagrams**. Officially tying fluid mechanics with electrical theory, if it wasn't clear already.

There are many other examples. And to investigate them is to learn. But, we are interested now in developing relations for our toolbox.

(Continues)

Suppose we start with a closed, *internal flow* circuit. Containing an *effectively incrompressible* fluid (we don't need this last one in general).

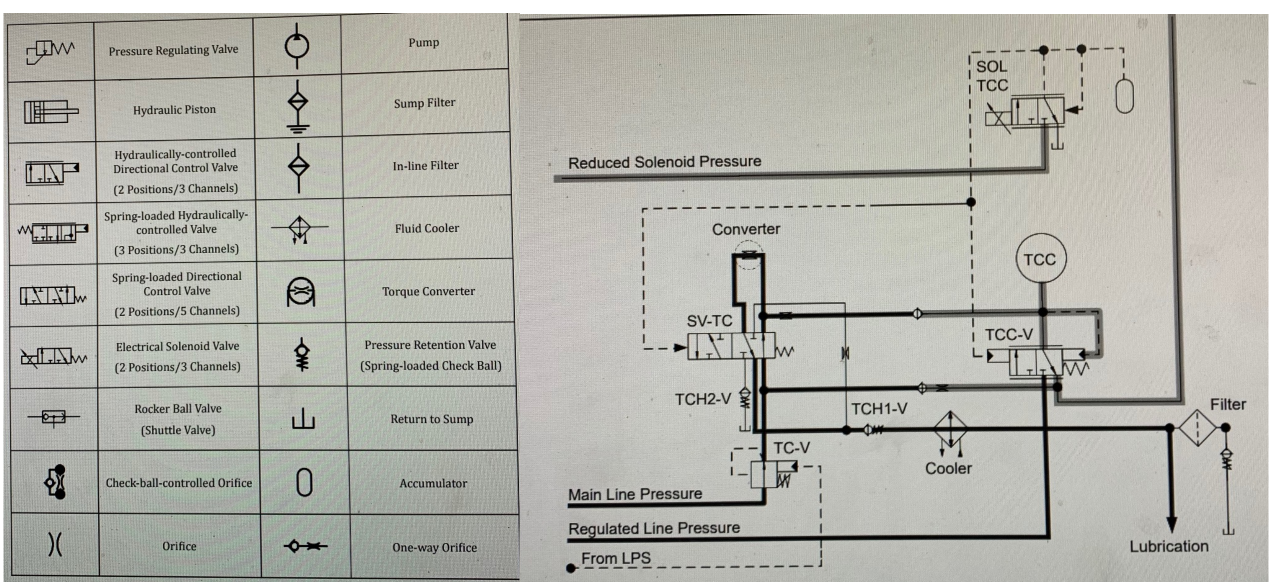


Shown on the right is some ideas of what we can do: introducing a pump, we can get the fluid moving. Further upstream we have a constriction, an expansion, a path splitting, and a gate/valve (switch). One can imagine state changers or other things that induce vortices or create secondary flows, etc.

Keeping it even more simple, say we just had the pump and the gate. If the gate is closed and the circuit is full of fluid, incompressibility should stall the pump (as we can't force more fluid into the post-segment.

However, if the gate was on one of the parallel segments, we can divert the stream to one side or the other. So theoretically, we could have a pressurized circuit with parallel options that only get "energized" when the gates are open. *We should take care before assuming that things act exactly the way electric circuits do.*

Just for a glance, I've listed an example pressure circuit and legend I found skimming.
Purposefully suppressing citation because you're not going to find it anyways.



(Continues)

The following is based on retrospect from the main text discussion (Ch.3, pg.49+ [3]), Sections 1.1 and 2.1 of this document, and otherwise.

Recall our definitions of pressure and density of a fluid, centered at a point in a volume:

$$\boldsymbol{p(x)} := \frac{\int_S (\boldsymbol{F} \cdot \boldsymbol{n}) d\boldsymbol{A}}{\int_S d\boldsymbol{A}}$$

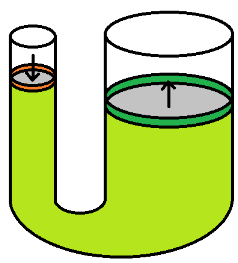
$$\boldsymbol{M}_{\text{in V}} =: \int_V \boldsymbol{\rho(x)} d\boldsymbol{V}$$

and the first Physical System Law:

$$\frac{d}{dt} \boldsymbol{M_{total}} = 0$$

In developing a free body diagram for a *submerged object*, one considers surface and body forces and analyzes the relation between associated integrals of the force fields to determine kinematics. Moreover continual application of forces brings into consideration work/energy/thermo.

Similarly, we can develop diagrams at the interfaces (free surfaces) for *open hydraulic circuits*. In the sense of say manometers. Similar as well, the fliud-solid interface between movable boundaries of a closed (liquid-sealed) hydraulic circuit.



If we combine equations 2 and 3, we get:

$$\frac{d}{dt} \boldsymbol{M}_{\text{in V}} = \frac{d}{dt} \int_V \boldsymbol{\rho(x)} d\boldsymbol{V} = 0$$

If $\boldsymbol{\rho}$ is constant throughout and time-invariant, this becomes:

$$\boldsymbol{\rho(x)} \frac{d}{dt} \boldsymbol{V} = 0$$

or equivalently:

$$\boldsymbol{V_{action}} = \boldsymbol{V_{reaction}}.$$

To connect with equation 1, note that we have ($\boldsymbol{A_{base}} = \boldsymbol{V_{cyl}/h}$), so that we may write:

$$(\boldsymbol{V_{action}})\boldsymbol{p_{action}(x)} = \boldsymbol{h_{action}} \int_S (\boldsymbol{F_{action}} \cdot \boldsymbol{n}) d\boldsymbol{A_{action}}.$$

(Continues)

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Consequently,

$$\boldsymbol{V_{action}} = \frac{\boldsymbol{h_{action}} \int_{\boldsymbol{S}} (\boldsymbol{F_{action}} \cdot \boldsymbol{n}) \boldsymbol{dA_{action}}}{\boldsymbol{p_{action}}(\boldsymbol{x})}.$$

and hence:

$$\frac{\boldsymbol{h_{action}} \int_{\boldsymbol{S}} (\boldsymbol{F_{action}} \cdot \boldsymbol{n}) \boldsymbol{dA_{action}}}{\boldsymbol{p_{action}}(\boldsymbol{x})} = \frac{\boldsymbol{h_{reaction}} \int_{\boldsymbol{S}} (\boldsymbol{F_{reaction}} \cdot \boldsymbol{n}) \boldsymbol{dA_{reaction}}}{\boldsymbol{p_{reaction}}(\boldsymbol{x})}.$$

If the direction of application is uniform and parallel with the normal, we simplify:

$$\frac{\boldsymbol{Ah|F|}}{\boldsymbol{p(x)}}_{\boldsymbol{action}} = \frac{\boldsymbol{Ah|F|}}{\boldsymbol{p(x)}}_{\boldsymbol{reaction}}$$

Reiterating here that $\boldsymbol{x}$ is a reference point, for the containing volume.

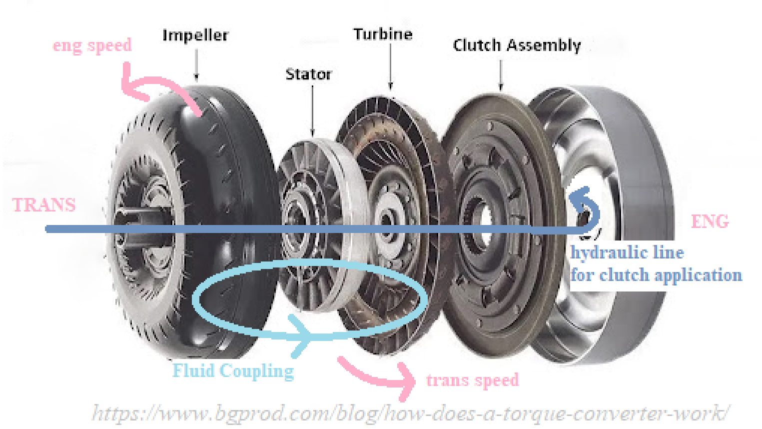
[Exercise: Repeat this argument without constraints on density and perhaps allow for variable force fields and piston application surfaces.]

[Exercise: Factor in the no-slip condition and consider boundary layers (Recall Section 1.4).]

(Continues)

5.4: Culmination

Short of implementation details (especially PCM strategies for hydraulically controlling gear selection with clutches) the only thing left that I want to cover is this mysterious torque converter operation, whose interest actually prompted the creation of this entire document.



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*The **engine** generates torque onto its **output shaft**. On the output shaft there is a **flywheel** (metal disc with holes).*

*Separately, the **transmission** has all of its rotating components on an **intermediate shaft** internal to the **housing**. It's **output shaft** goes to the **driveshaft** etc.*

*In between the engine and trans lies the **torque converter**. The TC is usually splined to the trans as male to female respectively. When putting the trans together with the engine, one usually bolts the TC onto the flywheel, subsequently bolting the trans casing "bell housing" to the engine casing.*

*Due to the fasteners, we see that the TC housing spins with the engine output. The TC is filled with fluid (closed system). The **impeller** is on the housing and is used to pump the fluid in a toroidal circuit in an "exothermic" manner, transferring energy from solid blades to the fluid.*

This TC housing contains, but is not attached to the internal components, which rotate with the shaft going into the transmission.

*This device is used to connect two spinning shafts with an intermediate, pressurized **fluid coupling** as opposed to splining or otherwise solid-solid connection.*

Note that gear selection is a distinct process from what is happening here. Though selection of gear determines functionality of this component (in the sense of clutch engagement).

There are two states: when the hydraulic clutch is applied, power flows through it from the engine to trans; when it is not applied, no power flows. There is a hybrid state, where the clutch is allowed a level of slipping, which causes partial flow of power. This all is to later be multiplied by the gearsets.

The Stator/Turbine combination is a Stage *that we saw previously. It operates in the "endothermic" manner, transferring energy from fluid to solid, but also rerouting the fluid to go back to the pump.*

In closing and locking the toolbox for the day, we see Axioms of Fluid Mechanics, the Physical System Laws, the Similitude Relations and the Equations of State once more... Bright and early next time!

■ END DOCUMENT ■

Further study ideas: Any of the in-text exercises/problems listed, watch the NCFMC videos, look into microprocessors and programming, create a real hydraulic or electrical circuit and test it. Explore the hydraulic circuit theory and electrical analogs. Maybe revisit diffusion of turbulence again. The phase diagram analysis was briefly touched...

(Continues)

Appendix (1): Phenomenology

The following is a list similar to that seen in Section 2.3. But the focus shifts to the phenomena, vs. the other was focusing on the experimental apparatuses used to measure the phenomena. There is a somewhat logical ordering:

⟨⟨ PHENOMENA ⟩⟩

1 Fluid Elements; Clusters/Control Volumes
2 Viscosity/Shear Stress; Inertial Forces
3 No Slip, Boundary Layers; and Lubrication
4 Movement and Vorticity/Vortices
5 Coriolis Forces; Waves
6 Compressibility; Force Transmission
7 Viscous Reversible Flows/Memory Fluid
8 Thermodynamic State Change
9 Cavitation
10 Turbulence; Energy Diffusion (Waterfall)
11 Choke, Instability, Shock Waves; Flow Regimes:
  > No Stall,
  > Large Transitory Stall,
  > Fully Developed Stall,
  > Hysterisis,
  > Jet Flow, etc.
12 Vibrational Modes
13 Buoyancy; Surface Geometry, and Drag
14 Surface Tension and Energy Minimalization; Flow Stratification
15 Magnetohydrodynamics

Analyzing this list, it seems the important topics for study are: forces applied to or between fluid elements and solids and the corresponding energy and thermodynamic considerations. The fact that shock waves can be visualized as discontinuities in pressure curves over manometer series is very interesting to me mathematically. Regularity of solutions of relevant equations seem to be allowed piece-wise smooth or less!?

List of Videos From Reference [21]:
"Aerodynamics Generation of Sound, Boundary Layer Control, Cavitation, Channel Flow of a Compressible Fluid, Deformation of Continuous Media, Eulerian Lagrangian Description, Flow Instabilities, Flow Visualization, Fluid Dynamics of Drag Part I, II, III, and IV, Fundamental Boundary Layers, Low Reynolds Number Flow, Magnetohydrodynamics, Pressure Fields and Fluid Acceleration, Rarefied Gas Dynamics, Rheological Behavior of Fluids, Rotating Flows, Secondary Flow, Stratified Flow, Surface Tension in Fluid Mechanics, Turbulence, Vorticity, Part 1; Vorticity, Part 2; and Waves in Fluids".

(Continues)

Appendix (2): Electrical Circuit Theory

The following is based off of [4, 14, 17] and prior experience.

Current, Voltage, and Impedance:

• Def: A **circuit** is a closed path of **insulated conducting** material. A **circuit segment** is just a portion of the path. And **circuit elements** we'll define as **electrical input/output devices**. Typical paths are along **wires** and **connectors** may be used to adjoin circuit segments by closing contact with the **terminals** from each. **Wiring diagrams** or **circuit diagrams** are useful for analysis and tracing of problems. Such are *weighted*, (ideally *directed*) *graphs*.

• Def: **Current** is thought of and referred to as the *flow* of free electrons through a circuit. But, the formal quantity is: the amount of charge flowing per second through an infinitesimal cross-section of relevant circuit wire. Units are **Coulombs/second**: $[\boldsymbol{i}] := \boldsymbol{C/s} =: \boldsymbol{A}$, which defines **Amperes**.

• Def: Circuit elements that change the **kinetic energy** of each of these charges are called **electromotive force (emf) sources**. A common example is a **battery**. [Exercise: Investigate the construction and operation of emf's.] The formal quantity of: kinetic energy applied per charge is called **Voltage**. Units are **Joules/Coulomb**: $[\boldsymbol{V}$ or $\boldsymbol{\varepsilon}] := \boldsymbol{J/C} =: \boldsymbol{V}$, which defines **Volts**.

• Def: Suppose the electrons in our circuit have a time-varying emf source,

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon(t)} \text{ (in Volts).}$$

If this function is constant in time, we have **Direct current (DC)**. If this function is oscillatory, we have the umbrella term of **Alternating current (AC)**. Other forms such as **square waves** (on/off DC with **duty cycles**), and more general **signals** exist. An important note is that *propagation of voltage/current is forward and back along the wire* and it is convenient to model as the real projection of complex wave functions:

$$\boldsymbol{\varepsilon} : [\boldsymbol{0}, \boldsymbol{\infty}) \to \mathbb{R}^{\boldsymbol{2}} \cong \mathbb{C};$$

$$\boldsymbol{\varepsilon(t) := A(t) \cdot e^{j\omega(t)} = a(t) + j \cdot b(t)},$$

with imaginary unit $\boldsymbol{j}$, *amplitude function*, $\boldsymbol{A}$, and *frequency function* $\boldsymbol{\omega}$ as well as Cartesian component functions $\boldsymbol{a(t)}$ and $\boldsymbol{b(t)}$. ***Remember to state conventions for coordinate systems.*

• Def: Lastly, there is a broad notion of electrons "resisting" the transfer of kinetic energy in a circuit element. With respect to the above special form of emf, an empirically linear relation between voltage and current gives the derived quantity of **Impedance**:

$$\boldsymbol{\varepsilon(t) \sim Z \cdot i(t)},$$

whose units are $[\boldsymbol{Z}] := \boldsymbol{V/A} =: \boldsymbol{\Omega}$, defining **Ohms** in "**Ohms Law**".

Related Theories:

Based on the discussion, it should be clear that the relation between voltage and current (in a circuit element) need not be linear. And so we would have to name a bunch of other coefficients, or as in the viscosity discussion (see Section 1.3), give some kind of *apparent impedance*, etc. Limiting the discussion to only the linear case, we study so called [Linear Network Analysis].

Further terms exist in this linear theory, such as: (**impedance** = **resistance** + $j\cdot$ **reactance**) = 1/(**admittance** = **conductance** + $j\cdot$ **susceptance**), **inductance**, **capacitance/elastance**. This is not even including the magnetic correspondents such as **reluctance**. But I am reluctant to elaborate on all these at the moment. Basically, the terms state real and imaginary components of impedance and there are reciprocal versions and there exist magnetic correspondents due to the Maxwellian-Gaussian-Faradayian-Amperian intertwining of the E and B fields, together with a plethora of stupid units.

The rest of the Linear Network Theory consists of calculating these values in **series** and **parallel** across various **graph/network topologies** and creating a notion of equivalence classes thereof, "$[circuit]_\sim$". [Exercise: Further investigate the sidebar in the wiki reference. Especially the Generator Theorems etc.] [Exercise: Read up on the development of Maxwell's equations in Halliday, as well as other important things we left out on say capacitors and inductors, that and more are all there.]

These are no doubt used in the creation of circuit elements we find in practical settings: *Hall effect* sensors, etc.

If you want to dive in to quantum electrodynamics/chromodynamics, relativity, etc. you'll find a number of things including: *spinors* and *twistors* (*phasors?*) that are mathematically interesting, as well as wave/particle duality, more conservation laws, etc.

Basic Diagnosis:

To finish the section let's talk about introductory practical diagnosis of circuit faults. This will include an overview of multimeter functions and basic circuit damage tracing.

• Def: An **open** in a circuit is a physical discontinuity in the wiring that does not allow current to cross. Visual inspection would suffice if one could see all the wiring. If this is not the case, one could instead check each segment of the circuit by probing the terminals of connectors and either testing for **Continuity** or **Load Testing**. See also **Voltage Measurement**.

Side note: **Resistors** can be used to change the current value for purposes of input to modules or circuit identification with *biases*. This is another reason to use the Ohm-meter.

(Continues)

• Def: A **short-circuit** happens when two wires' conductive material make contact when they are not supposed to. This effects the path of the current in potentially unexpected ways, leading to improper operation of modules, etc. Again, visual inspection may suffice to find where this is happening, but if not we must infer through voltage/current/resistance readings. If values are known, observe deviations from the known values. If two wires meet, their readings *may* be averaged. This algebraic indicator *could* tell you which wires are meeting. This goes for shorts to **power**, **ground**, etc.

Side note: Conventionally, current orientation is outbound from the "+"-terminal of a battery. But, the electrons are negative charges and flow the other way. That being said, in terms of flow, "power" refers to the circuit segment coming from the positive side. "Ground" is still kind of a mystery to me, but refers to the segment that connects to the negative terminal of the battery directly or indirectly (e.g. via chassis). [Exercise: What is a ground?]

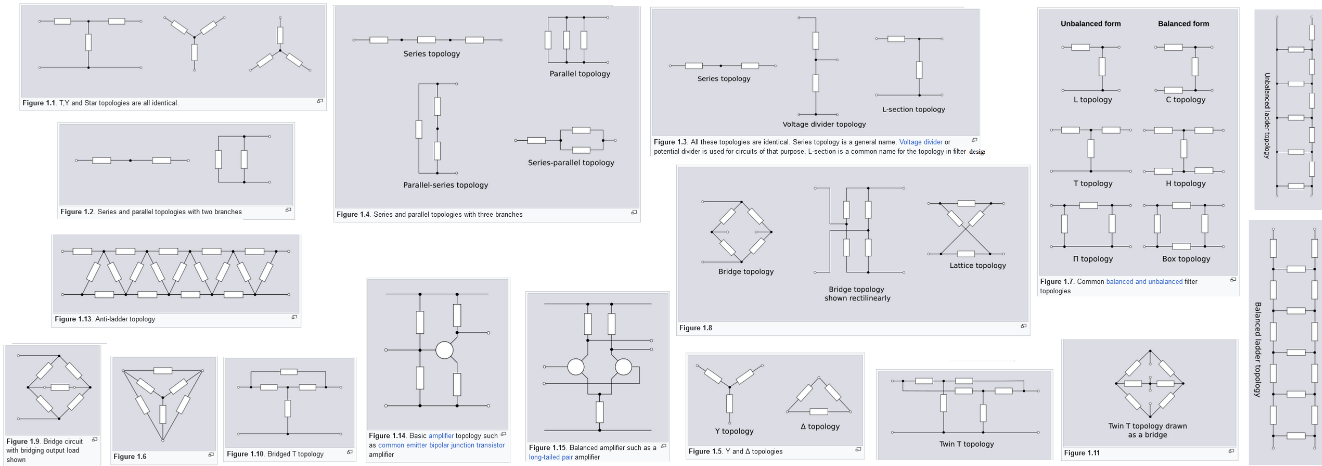
Side note (2): A single wire may be purposefully **spliced** into multiple (fibers divided) or have multiple wires be spliced into one (wires **soldered/sautered** together). This is not a short, but rather a parallel circuit consideration. Division of fibers would cause a change in resistance though and different **gauges** have different resistances. [Exercise: Look into this.]

• Def: A **draw** (a.k.a. parasitic draw) is a condition where a battery is being drained by modules that won't *switch* off for some reason even though they are commanded off. This could be due to shorts somewhere leading to incorrect inputs to a module, making a *relay* operate incorrectly for example. In the case for automotive, DTC's are useful in the hunt.

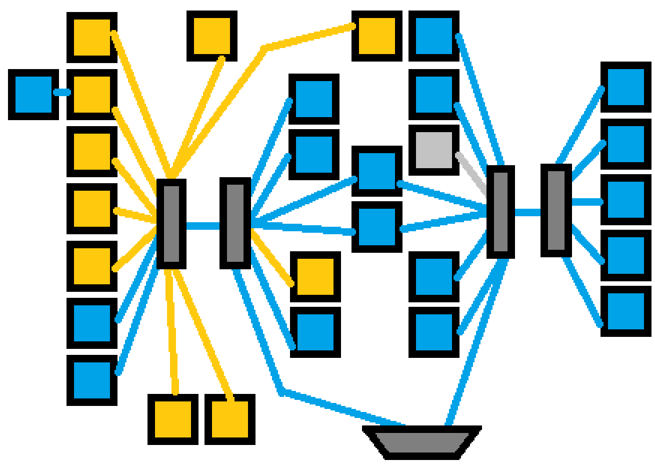
Some Multimeter Functions:
• REL Δ - Allows for readings relative to a set value.
• Data Hold - Screenshot essentially.
• NCV - Non contact voltage reading (likely magnetic inference).
• Bar Graph - Display function.
• AC/DC Voltage Measurement
• Current Measurement
• Frequency/Duty Cycle Measurement
• Impedance Measurement
• Resistance Measurement
• Diode Measurement - Measures voltage drop across it.
• Capacitance Measurements - Charges a capacitor, measures resulting voltage etc.
• Temperature Measurement

(Continues)

Further Enlightenment

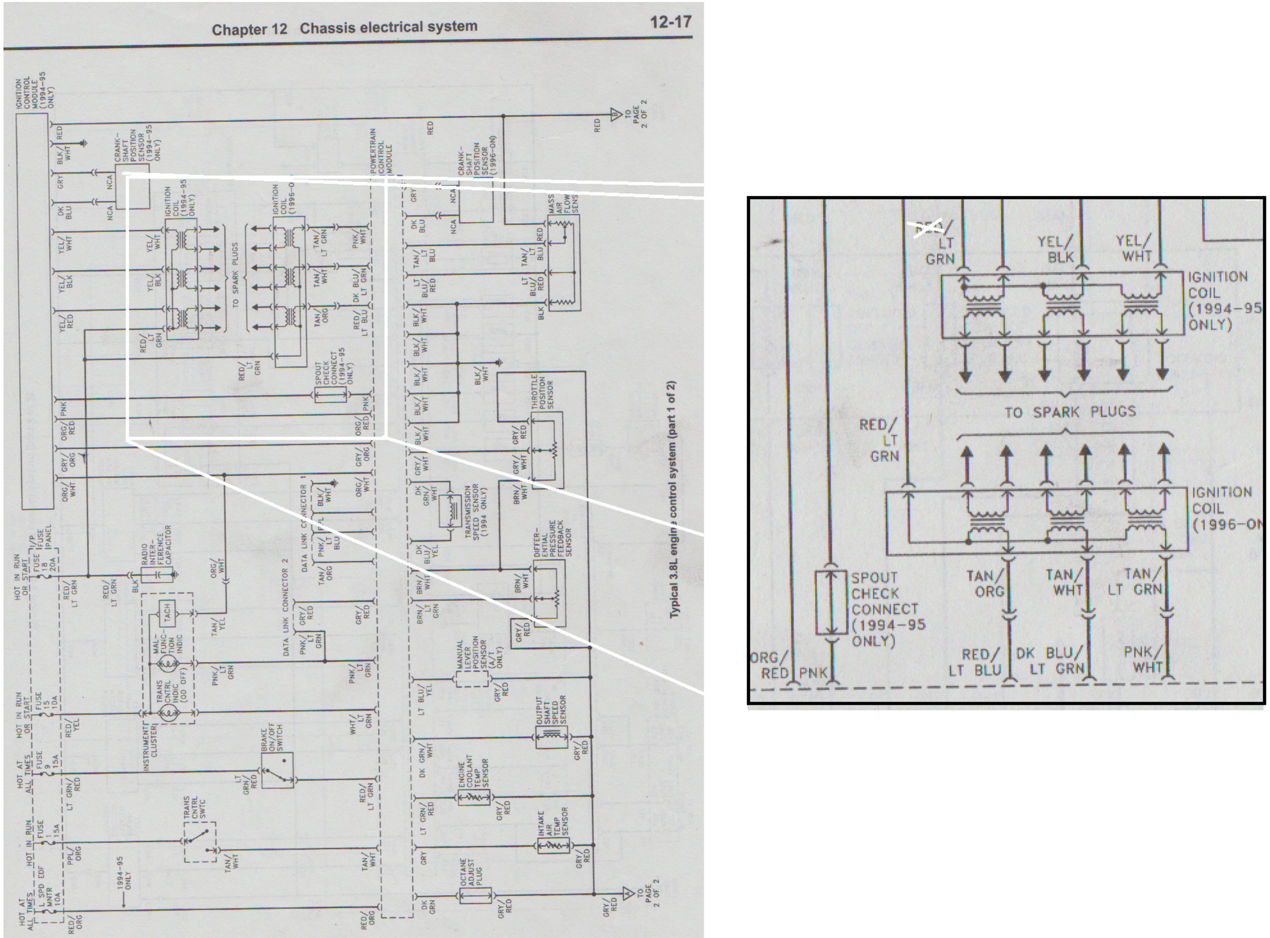


Collected from the article [12]. These are used for various purposes.



Abstracted from a *Stellantis Topology* found on the google. This is an example of an actual final product with modules such as Powertrain Control Module, HVAC, BCM, etc. Included also are SGW and some Star connectors. We should distinguish here that the "Topology" is a *module communications graph*, it does not show power/ground etc. **Wiring buses** are differentiated by colors and you can get creative otherwise.

(Continues)



Taken from a random *Ford Hayne's Repair Manual*. This looks inside some of the modules and has weights on the wiring for at least *wire coloring* (ex: "DK BLU/LT GRN"). Modern stuff shows **connector ID's**, **circuit segment ID's**, and also **terminal pin numbers** within the connectors, for aid in diagnostics. It is useful going forward to look up different circuit element symbols (resistor etc.) and make a legend for yourself.

In Closing

How can we tie this back in to fluid mechanics? Make analogies with batteries as fluid pumps, voltage as pressure more or less and damage diagnostics with leaks are a thing. However, state changes with HVAC components (compressors, condensors, etc.) are additional players not really seen at the surface level in electrical theory. Changing the state of electrons aside from kinetic energy... to be continued.

MISC POST NOTES:
To measure signals etc. an oscilloscope must be used.

(Continues)

References

[1] Alligood, Kathleen T. (1996). *Chaos: An Introduction to Dynamical Systems*. Springer.

[2] Evans, Lawrence C. (2010). *Partial Differential Equations (Second Edition)*. American Mathematical Society.

[3] Fox, Robert W. and McDonald, Alan T. (1985). *Introduction to Fluid Mechanics (Third Edition)*. John Wiley & Sons.

[4] Halliday, David et al. (2002). *Physics Volume 2 (5th Edition)*. John Wiley & Sons.

[5] Hollabaugh, Mark (2013). *Physics [Course 3 in associate sequence]*. Pearson Custom Library.

[6] Lee, John M. (2013). *Introduction to Smooth Manifolds (Second Edition)*. Springer.

[7] White, Frank M. (?yr). *Fluid Mechanics (4th Edition)*.

> **Internet References:**

[8] Aero & Air. (03.31.2021) . Youtube Video:
"https://www.youtube.com/watch?v=pjLaFkIDZBk."

[9] Wikipedia contributors. Conservation law. Wikipedia, The Free Encyclopedia. January 10, 2024, 13:06 UTC. Available at:
"https://en.wikipedia.org/w/index.php?title=Conservation_law&oldid=1194731057."

[10] Wikipedia contributors. Continuity equation. Wikipedia, The Free Encyclopedia. June 11, 2024, 10:36 UTC. Available at:
"https://en.wikipedia.org/w/index.php?title=Continuity_equation&oldid=1228461211."

[11] Wikipedia contributors. Continuum mechanics. Wikipedia, The Free Encyclopedia. August 11, 2024, 11:40 UTC. Available at:
"https://en.wikipedia.org/w/index.php?title=Continuum_mechanics&oldid=1239764820."

[12] Wikipedia contributors. Circuit topology (electrical). Wikipedia, The Free Encyclopedia. October 19, 2024, 05:19 UTC. Available at:
"https://en.wikipedia.org/w/index.php?title=Circuit_topology_(electrical)&oldid=1251985977."

[13] Wikipedia contributors. Dynamical system. Wikipedia, The Free Encyclopedia. June 8, 2024, 22:55 UTC. Available at:
"https://en.wikipedia.org/w/index.php?title=Dynamical_system&oldid=1227998017."

[14] Wikipedia contributors. Network analysis (electrical circuits). Wikipedia, The Free Encyclopedia. July 23, 2024, 22:53 UTC. Available at:
"https://en.wikipedia.org/w/index.php?title=Network_analysis_(electrical_circuits)&oldid=1236294816."

[15] gurelyasin. (03.29.2007) *No Slip Condition*. Youtube Video: "https://www.google.com/search?client=firefox-b-1-d&sca_esv=591469718&q=no+slip+condition+experiment&tbm=vid&source=lnms&sa=X&ved=2ahUKEwiL2Ir93pODAxWiPUQIHaXaDYsQ0pQJegQIDRAB&biw=1920&bih=927#fpstate=ive&vld=cid:29722380,vid:cUTkqZeiMow,st:0".

[16] Wikipedia contributors. Reynolds transport theorem. Wikipedia, The Free Encyclopedia. April 22, 2024, 05:37 UTC. Available at: "https://en.wikipedia.org/w/index.php?title=Reynolds_transport_theorem&oldid=1220165017."

[17] gandalf61 (https://physics.stackexchange.com/users/148491/gandalf61). "Q/A: Electrical Circuits - How does AC voltage/current propogate?". URL (version: 2024-12-08): "https://physics.stackexchange.com/q/836538."

[18] American Society of Heating, Refrigerating and Air-Conditioning Engineers (2009). *Thermophysical Properties of Regrigerants*. ASHRAE. "https://www.researchgate.net/profile/Praveen-Alok/post/Where-can-I-get-the-property-chart-for-viscosity-and-thermal-conductivity-for-R245fa-and-R1233zde/attachment/5c6c0939cfe4a781a5814ba6/AS%3A728016720232452%401550584120700/download/SI_F09_Ch30.pdf."

[19] National Center for Biotechnology Information. Periodic Table of Elements. URL: "https://pubchem.ncbi.nlm.nih.gov/periodic-table/". Accessed Jan. 23, 2025.

[20] PVT Surface. Princeton University Website. URL: "https://www.princeton.edu/ humcomp/sophlab/ther_4.htm". Accessed Jav. 23, 2025.

[21] National Committee For Fluid Mechanics Films (a.k.a. "NCFMC"). *Entire Filmography*. Referenced in duration of this project 2024+. Available at: "http://web.mit.edu/hml/ncfmf.html."

[22] Fefferman, Charles L. (Accessed 02.22.2025) *Existence and Smoothness of the Navier Stokes Equation*. Clay Institute Website. "https://www.claymath.org/wp-content/uploads/2022/06/navierstokes.pdf."

[23] Wikipedia contributors. Turbomachinery. Wikipedia, The Free Encyclopedia. April 30, 2024, 18:52 UTC. Available at: "https://en.wikipedia.org/w/index.php?title=Turbomachinery&oldid=1221576473."