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Introduction

What is this paper about?

The purpose of this work was originally to *consolidate theory* in the branches of *Real Analysis* and *Topology* to add to an ongoing collection of *par-grad-level* material–which, at the time of writing, included: Abstract Algebra, Differential Geometry, Logic, and Category Theory (available on my website located HERE). As this paper developed, areas for relative further research were discovered and appended as described below and in the table of contents.

What does it encapsulate?

The theoretical lenses named above allow us to focus on constructions involving functions of different types $\boldsymbol{f : X \to Y}$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are taken to range from *product spaces* of $\mathbb{R}$, to metric, topological, and measure spaces and potentially more. Understanding such constructions requires at least the notions of sequences, limits, convergence, boundedness, suprema and infima. Proving assertions made in the constructions requires logical manipulation and certain analytical techniques to be seen in the document.

Throughout the writing process and with retrospect applied, critical proof points for the theory were embedded and appended in the Results section when they were too long to be embedded in the discussion. I have made it a point to unearth as many details as possible without being pedantic. As a gauge of sophistication level for the par aforementioned, I have included and separately referenced two comprehensive exams.

So, the topics encapsulate those met on the exams, as well as general theory from classes I have taken before and from new texts I have come across.

What does it leave out?

We stick to core definitions and theory, things like the Mean Value Thoerem, L'Hospital's Rule, formulas for derivatives, the Chain Rule, Implicit and Inverse Function Theorems, and other main stream theorems like Egorov's Thoerem are left out unfortunately. The advanced topics one comes across in the field like fractal measures and transcendental cardinalities are not covered here. The hope is that what is contained here is sufficient to push the $\boldsymbol{\partial}$ to those topics after the fact.

What unique features are there?

The "Read Me" link on the title page should be of some use. It describes the structure and the formatting graphically. Although the paper in print can be read linearly and navigated traditionally, the PDF version can be referenced non-linearly with easy to use link structure and color coating forms of information make digestion a little easier as well (in my opinion). Additionally, the citations, if not listed in the beginning of each section or subsection, permeate the content where applicable.

Special thanks to my professors in Real Analysis I-II and Topology (Richard Shoen's 140A/B and Svetlana Jitomirskaya's 141 at UCI $\sim$ 2016), and Algebraic Topology (Julian Paupert's 501 at ASU $\sim$ 2018). Special thanks to my brother Justin for being my discussion partner in a time of great mathematical solitude in 2021.

1. Preliminaries in Abstract Algebra

See references ([6],[7],[10]) for more on these subjects.

1.1: Algebraic Structures

• Def An **algebraic structure**, $\mathfrak{A}$, consists of: a **set/class** $\boldsymbol{A}$, a (finite) collection of **operations/functions**, $\{f_i\}_{i\in I}$ of respective **arities** $\tau_1 = (m_i)_{i\in I}$, and a (finite) collection of **axioms** in the form of **relations** $\{R_j\}_{j\in J}$ of respective arities ($\tau_2 = (n_j)_{j\in J}$). Summarizing:

$$\mathfrak{A} := \Big\{A,\ \{f_i\}_{i\in I},\ \{R_j\}_{j\in J},\ \tau\Big\}.$$

Where the **type** of the algebra is the union of all the arities: $\tau := \tau_1 \cup \tau_2$.

Some main examples of algebraic structures include *groups*, *rings*, *fields*, *vector spaces*, *modules*, and *algebras*. There are examples that deviate from the main ones in terms of relaxing or adding axioms, such as: *semi-groups*, *monoids*, *division rings*, etc.. More complete listings can be found online. The ones we care about in this paper are the main ones up through vector spaces. We will also encounter *metric spaces* and *topological spaces*, *posets* and various *ordered structures*. We'll start on the next page.

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• Def A **group** is a set, G together with operations $\{*, (\cdot)^{-1}, 1_G\}$ that have arities $(2, 1, 1)$. The operations can be given aliases, here we have "product", "inversion", and "identity", the latter two of which are bijective as functions. We stipulate the relations:

$$\forall g \in G,\ g^{-1} * g = g * g^{-1} = 1_G$$

$$\forall g \in G,\ 1_G * g = g * 1_G = g.$$

Qualifying further with the term **abelian** or **commutative** adds the relation:

$$\forall g, h \in G,\ g * h = h * g.$$

• Def A **ring** is a set R, together with two operations $*, +$, two respective inversion operations and an additive identity operation 0_R, so the type would be $(2, 2, 1, 1, 1)$. The "multiplicative" inversion operation is not required to be bijective but the other one is. Thus the stipulations are that $(R, +, (\cdot)_+^{-1}, 0_R)$ forms an *abelian group* and that the operations $(+, *)$ are compatible:

$$\forall r, s, t \in R,\ r * (s + t) = r * s + r * t$$

$$\forall r, s, t \in R,\ (r + s) * t = r * t + s * t.$$

Furthermore, we call a **ring with unit**, one such that there exists a multiplicative identity, 1_R. The structure $(R, *, (\cdot)_*^{-1}, 1_R)$ may or may not form a group. And there are different names for rings with added conditions ("commutative ring", "division ring", etc.)

In particular...

• Def A **field**, F is a *unital ring*:

$$\mathfrak{F} = \Big\{F,\ \{+, *, (\cdot)_+^{-1}, (\cdot)_*^{-1}, 0_F, 1_F\},\ \{R_i\}_{i\in I}, \{(2, 2, 1, 1, 1, 1), (\ldots)\}\Big\}$$

where the relations/stipulations $\{R_i\}_{i\in I}$ are that both $(F, +, 0_F)$ and $(F \backslash \{0_F\}, *, 1_F)$ give rise to *abelian groups.*

• Def Given a field F, a **vector space over F**, denoted $V = V/F$ is an *abelian group* $(V, +, 0_V)$, together with a **scalar action** $* : F \times V \to V$, also satisfying *distribution* of $(*/+)$ and $(+/*)$. As a structure, it looks like:

$$\mathfrak{V} = \Big\{F \amalg V,\ \{+, *, (\cdot)_+^{-1}, 0_V, 1_F\},\ \{R_i\}_{i\in I},\ \{(2, 2, 1, 1, 1), (\ldots)\}\Big\}$$

The term "action" carries axioms with it, namely *associativity* and *unit action*:

$$(\lambda \tilde{*} \mu) * v = \lambda * (\mu * v) \qquad \text{and} \qquad 1_F * v = v$$

for arbitrary $\lambda, \mu \in F,$ and $v \in V$. Further, the action is two-sided ($\lambda * v = v * \lambda$).

We've covered groups through vector spaces. Let's do metric/topological spaces next.

• Def A **metric space** is a set/class $\boldsymbol{M}$, together with a (real) binary functional:

$$\boldsymbol{d : M \times M \to \mathbb{R}}$$
$$\boldsymbol{(m, n) \mapsto d(m, n)}$$

satisfying the following:

(i) $\forall \boldsymbol{m, n \in M,\ d(m, n) = d(n, m)}$, [Symmetry]

(ii) $\forall \boldsymbol{m, n \in M,\ d(m, n) \geq 0}$ with equality only when $\boldsymbol{m = n}$, and [Positive-Definiteness]

(iii) $\forall \boldsymbol{m, n, p \in M,\ d(m, n) + d(n, p) \leq d(m, p)}$ [Triangle Inequality]

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As an algebraic structure, we have:

$$\mathfrak{M} = \Big\{\boldsymbol{M \amalg \mathbb{R},\ \{d\}, \{R_i\}_{i\in I},\ \{(2), (...)\}}\Big\}$$

where of course (i)-(iii) constitute the relations and since we've adjoined the reals to the underlying space, the operation/functional, $\boldsymbol{d}$, is inherently closed in $\mathfrak{M}$. In practice, metric spaces are usually denoted simply by $\boldsymbol{(M, d)}$.

• Def Given a set $\boldsymbol{X}$, denote the **power set of** $\boldsymbol{X}$ by $\mathscr{P}(\boldsymbol{X})$. The power set is the collection of ALL subsets of $\boldsymbol{X}$ (including the empty set, $\emptyset$). Take a subset $\widetilde{\boldsymbol{T}} \subset \mathscr{P}(\boldsymbol{X})$ and close it under **finite intersection**, **arbitrary union**, and append elements $\boldsymbol{\emptyset, X}$. After this closure, rename the set $\boldsymbol{T} := \langle\widetilde{\boldsymbol{T}}\rangle$. We call the pair $\boldsymbol{(X, T)}$ a **topological space**.

Again, we list the axioms for the **topology**, $\boldsymbol{T}$:

(i) $\boldsymbol{\emptyset, X \in T}$,

(ii) For $\boldsymbol{n < \infty,}$ we have $\forall \boldsymbol{S_1, ..., S_n \in T,\ \bigcap_{i=1}^{n} S_i \in T}$, and

(iii) For indexing set $\boldsymbol{I}$, $\forall \boldsymbol{S_i \in T,\ \bigcup_{i\in I} S_i \in T}$.

We declare the elements $\boldsymbol{S \in T}$ to be called "open sets". We'll discuss this in Section 7.

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As an algebraic structure, we list:

$$\mathfrak{X} = \Big\{\boldsymbol{T \subseteq \mathscr{P}(X),\ \{\cap, \cup\},\ \{R_i\}_{i\in I},\ \{(2, 2), (...)\}}\Big\}$$

where again (i)-(iii) constitute the relations. The set operations apply to other structures as well, but we require closure here, so it becomes relevant in the presentation of topological spaces.

1.2: Relations, Equivalences; Orderings, Bounds, Inf/Sup

• Def An **n-ary relation on a set S** is a subset of the n-fold cartesian product:

$$R \subseteq \underbrace{S \times \ldots \times S}_{n}$$

• Def We have special names for some *binary* relations satisfying individually:

(i) $\forall s \in S,\ (s,s) \in R$ [Reflexive]

(ii) $\forall s,t \in S,\ (s,t) \in R \leftrightarrow (t,s) \in R$ [Symmetric]

(iii) $\forall s,t \in S,\ \big((s,t) \in R \wedge (t,s) \in R\big) \leftrightarrow (t = s)$ [Anti-symmetric]

(iv) $\forall s,t,r \in S,\ \big((s,t) \in R \wedge (t,r) \in R)\big) \implies (s,r) \in R.$ [Transitive]

These will be building blocks (as axioms) for some more advanced relations on the next page.

• <u>Def</u> A binary relation is called an **equivalence relation** if (i),(ii), and (iv) hold above. Equivalence relations are sometimes denoted by the tilde symbol $\sim := R$ to remind us of their properties. Suppose $x \in S$. We define:

$$[x]_\sim := \{y \in S \mid (x, y) \in \sim\}$$

and call it the **equivalence class of x w.r.t $\sim$**.

The collection of all equivalence classes of a set with respect to an equivalence relation,

$$S/\sim := \{[x]_\sim \mid x \in S\}$$

is referred to as the **quotient of S by $\sim$**.

• <u>Def</u> When S is the underlying set of an algebraic structure, and the equivalence relation $\sim$ is "compatible" with the operations of the structure, the quotient becomes an algebraic structure of the same type. In this event, we call the relation a **congruence relation**, sometimes replacing the symbol $\equiv \leftrightarrow \sim$. By "compatibility", we mean the operations of the algebra are *natural* with respect to the equivalence classes (e.g. $[x * y] = [x]\widetilde{*}[y]$).

Lastly, we will need to discuss *orderings*, which are special types of relations used to put elements of a set in order (woah!). This may seem trivial, but it is crucial for the proofs we will encounter to define these relations properly and collect some of their properties.

• <u>Def</u> A **partial order**, $\leq := R$ is a *binary* relation on a set S such that (i), (ii), and (iii) from above are satisfied. That is, the relation is **reflexive**, **anti-symmetric**, and **transitive**.

[Exercise: Write out (i)-(iii) in terms of the $\leq$ symbol, it will jump out at you what we're up to. You might need the so called in-fix notation: $xRy \leftrightarrow (x, y) \in R$.]

A **total order**, is a partial order, where every two elements in S are **comparable**. That is:

$$\forall s, t \in S, \text{ either } (s, t) \in \leq \text{ or } (t, s) \in \leq .$$

• <u>Def</u> A **partially ordered set** or **poset**, P is simply a set with a partial order defined on it. We denote this by $(P, \leq)$. [Exercise: Write the algebraic structure form as we have done previously for groups through vector spaces, metric, and topological].

(Continues)

• <u>Def</u> Given a *poset*, $(\boldsymbol{P}, \leq)$, we make some more definitions regarding the elements and the relation:

For a subset $\boldsymbol{S} \subseteq \boldsymbol{P}$, an element $\boldsymbol{x} \in \boldsymbol{P}$ is an **upper bound** (resp. **lower bound for** $\boldsymbol{S}$) if it is such that $\forall \boldsymbol{s} \in \boldsymbol{S}, \boldsymbol{s} \leq \boldsymbol{x}$ (resp. $\boldsymbol{x} \leq \boldsymbol{s}$).

Further, the **least upper bound** for a subset $\boldsymbol{S} \subseteq \boldsymbol{P}$ is called the **supremum** and denoted $\boldsymbol{sup(S)}$. Similarly, the **greatest lower bound** is called the **infimum** and is denoted by $\boldsymbol{inf(S)}$. Where of course, $\boldsymbol{x} \leq \boldsymbol{y}$ means $\boldsymbol{x}$ is "less than or equal to" $\boldsymbol{y}$ etc.

• <u>Def</u> (Motivated by p.13-15 [15]) Suppose $(\boldsymbol{R}, +, *, \boldsymbol{0}, \boldsymbol{1})$ is a *unital ring* and that '$\leq$' is a *total order* imposed on the underlying set $\boldsymbol{R}$ such that the following hold for all $\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{\lambda} \in \boldsymbol{R}$:

1.) $(\boldsymbol{r} \leq \boldsymbol{s}) \implies ((\boldsymbol{r} + \boldsymbol{\lambda}) \leq (\boldsymbol{s} + \boldsymbol{\lambda}))$ [Translation]

2/3.) $[(\boldsymbol{0} \leq \boldsymbol{\lambda}) \wedge (\boldsymbol{r} \leq \boldsymbol{s})] \implies [(\boldsymbol{\lambda} * \boldsymbol{r}) \leq (\boldsymbol{\lambda} * \boldsymbol{s})$ and $(\boldsymbol{r} * \boldsymbol{\lambda}) \leq (\boldsymbol{s} * \boldsymbol{\lambda})]$
[Positive (Left/Right) Scaling]

then we say $\boldsymbol{R}$ is a **totally ordered ring**. In the case of a field, (2) and (3) are the same. We then refer to it as a **totally ordered field**.

These basic assumptions (1)-(3) imply other useful algebraic properties of the ordering. We will deduce them when necessary. This concludes the preliminaries section.

To summarize: we have defined algebraic structures in general, we've defined instances of algebraic structures including groups, rings, fields, and vector spaces. We've defined equivalence relations, congruence relations, partial and total orders, and totally ordered rings/fields (among other things).

In the sequel we will use our a priori knowledge of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ to postulate the type of structure that exists on each, then defining them axiomatically, we will be able to prove our assertions and will facilitate further theory involving functions etc.

2. Axiomatic Development of the Reals $\mathbb{R}$

• Def (p.1-2 [15]): The **Naturals, $\mathbb{N}$,** are given by the counting numbers with addition defined on them and a special least element $\mathbf{1}$. We additionally declare the so called *Peano Axioms*:

$\mathbf{1.)}$ $\mathbf{1} \in \mathbb{N}$ [Unit Element]

$\mathbf{2.)}$ $\forall \boldsymbol{n} \in \mathbb{N}, \exists \boldsymbol{m} \in \mathbb{N},$ such that $\boldsymbol{m} = \boldsymbol{n} + \mathbf{1}$ [Containment of Successors]

$\mathbf{3.)}$ $\forall \boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}, (\boldsymbol{m} + \mathbf{1} = \boldsymbol{n} + \mathbf{1}) \implies (\boldsymbol{m} = \boldsymbol{n})$ [Successors and Equality]

$\mathbf{4.)}$ $\forall \boldsymbol{m} \in \mathbb{N}, \mathbf{1} \neq \boldsymbol{m} + \mathbf{1}$ [Least Element]

$\mathbf{5.)}$ $\forall \boldsymbol{S} \subseteq \mathbb{N}, \Big((\mathbf{1} \in \boldsymbol{S}) \wedge (\forall \boldsymbol{n} \in \boldsymbol{S}, \boldsymbol{n} + \mathbf{1} \in \boldsymbol{S})\Big) \implies (\boldsymbol{S} = \mathbb{N})$ [Induction]

Note: The naturals $(\mathbb{N}, +, \mathbf{1}, \leq)$ don't even form a group, but they do form a totally ordered set [Exercise: Prove this! Define the relation $(\boldsymbol{m} \leq \boldsymbol{n})$.].

• Def (p.6 [15]): The **Integers, $\mathbb{Z}$,** are the set of positive and negative counting numbers, with two operations $(+/*)$ and two special elements $\mathbf{0}, \mathbf{1}$. Axioms consist of the following:

$\mathbf{1.)}$ $\mathbb{N} \subseteq \mathbb{Z}$. [Contains Naturals]

$\mathbf{2.)}$ $\exists(-\mathbf{1}) \in \mathbb{Z}$ and $\forall \boldsymbol{z} \in \mathbb{Z}, \ -\mathbf{1} * \boldsymbol{z} \in \mathbb{Z}$. [Append Negatives]

$\mathbf{3.)}$ $\exists(\mathbf{0}) \in \mathbb{Z}$ and $\forall \boldsymbol{z} \in \mathbb{Z}, \ (\mathbf{0} * \boldsymbol{z} = \mathbf{0}) \wedge (\boldsymbol{z} + \mathbf{0} = \boldsymbol{z})$ [Append Zero]

$\mathbf{4.)}$ $\forall \boldsymbol{z} \in \mathbb{Z}, \ (-\mathbf{1} * \boldsymbol{z}) + \boldsymbol{z} = \mathbf{0}$ [Additive Inverse Condition].

• Prop: The integers $(\mathbb{Z}, +, *, \mathbf{0}, \mathbf{1}, \leq)$ form a *totally ordered, commutative ring with unit.* [Exercise: Prove this!]

(Continues)

• <u>Def</u> (6-12 [15]): The **Rationals, $\mathbb{Q}$,** are given by essentially the set of all *reduced* fractions made from elements of $\mathbb{Z}$, where the denominator is nonzero of course. They come equipped with two operations $(+/*)$ which we will define shortly, as well as a natural identification of $\mathbb{Z}$ with the subset of fractions whose denominator is 1. Let's be rigorous:

1.) Define $\boldsymbol{Q} := \mathbb{Z} \times (\mathbb{Z}\backslash\{\mathbf{0}\})$, together with operations defined via:

$$(\boldsymbol{a}, \boldsymbol{b}) * (\boldsymbol{c}, \boldsymbol{d}) = (\boldsymbol{a} * \boldsymbol{c}, \boldsymbol{b} * \boldsymbol{d}) \qquad \text{[a.k.a. } \tfrac{a}{b} \cdot \tfrac{c}{d} = \tfrac{a*c}{b*d}\text{]}$$

$$(\boldsymbol{a}, \boldsymbol{b}) + (\boldsymbol{c}, \boldsymbol{d}) = (\boldsymbol{a} * \boldsymbol{d} + \boldsymbol{b} * \boldsymbol{c}, \boldsymbol{b} * \boldsymbol{d}). \qquad \text{[a.k.a. } \tfrac{a}{b} + \tfrac{c}{d} = \tfrac{a*d+b*c}{b*d}\text{]}$$

2.) Now suppose $\boldsymbol{q} = (\boldsymbol{q_1}, \boldsymbol{q_2}),\ \boldsymbol{r} = (\boldsymbol{r_1}, \boldsymbol{r_2}) \in \boldsymbol{Q}$. Then define the relation:

$$(\boldsymbol{q} \sim \boldsymbol{r}) \leftrightarrow \Big(\exists \boldsymbol{\lambda} \in \mathbb{Z}\backslash\{\mathbf{0}\},\ (\boldsymbol{q_1} = \boldsymbol{\lambda} * \boldsymbol{r_1}) \wedge (\boldsymbol{q_2} = \boldsymbol{\lambda} * \boldsymbol{r_2})\Big).$$

[a.k.a. $\boldsymbol{q} \sim \boldsymbol{r} \leftrightarrow \frac{q_1}{q_2} = \frac{\lambda * r_1}{\lambda * r_2}$]

Then, observing that $\sim$ is an *equivalence relation* on $\boldsymbol{Q}$ [Exercise: Prove this!], we define:

$$\mathbb{Q} := \boldsymbol{Q}/\sim \; = \{[\boldsymbol{q}]_\sim \mid \boldsymbol{q} \in \boldsymbol{Q}\},$$

together with the *induced operations* on the *equivalence classes*.

<u>Note:</u> We may thus identify $\boldsymbol{z} \in \mathbb{Z}$ with $[(\boldsymbol{z}, \mathbf{1})]_\sim \in \mathbb{Q}$. This also includes the treatment of zero. [Exercise: Prove $\forall \boldsymbol{q} \in \mathbb{Q},\ \mathbf{0} * \boldsymbol{q} = \mathbf{0}$. Also try the other trivial identities.]

• <u>Prop:</u> The rationals $(\mathbb{Q}, +, *, \mathbf{0}, \mathbf{1}, \leq)$ form a *totally ordered field*. [Exercise: Prove this!]

(Continues)

So far, we have the inclusion chain $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$ and we wish to append $\mathbb{R}$ building off of the axioms of $\mathbb{Q}$ in such a way that we get another *totally ordered field.* We will prove that our declaration below gives us this.

• Def (12-29 [15]),([4, 5]):
The **Reals,** $\mathbb{R}$, can be axiomatized by the following:

1.) Let $\boldsymbol{\alpha} \subseteq \mathbb{Q}$ be a subset subject to (i)-(iii) below:

(i) $\emptyset \neq \boldsymbol{\alpha} \neq \mathbb{Q}$ [Proper Subset of $\mathbb{Q}$]

(ii) $\forall \boldsymbol{a} \in \boldsymbol{\alpha},\ \forall \boldsymbol{b} \in \mathbb{Q},\ (\boldsymbol{b} < \boldsymbol{a}) \implies (\boldsymbol{b} \in \boldsymbol{\alpha})$ [No Lower Bound] [a.k.a. $\boldsymbol{inf}(\boldsymbol{\alpha}) = -\infty$.]

(iii) $\forall \boldsymbol{a} \in \boldsymbol{\alpha},\ \exists \boldsymbol{b} \in \boldsymbol{\alpha},\ \boldsymbol{a} < \boldsymbol{b} < \boldsymbol{sup}(\boldsymbol{\alpha})$ [No Largest Rational]

then we say $\boldsymbol{\alpha}$ is a **Dedekind cut** (first declared by Richard Dedekind).

These subsets are designed to be *rational intervals* $\boldsymbol{\alpha} = (-\infty, \boldsymbol{r}) \cap \mathbb{Q}$, for $\boldsymbol{r} \in \mathbb{R}$, just stated in terms of $\mathbb{Q}$. Notice that $\boldsymbol{r} = \boldsymbol{sup}(\boldsymbol{\alpha})$ so that we identify real numbers with these prescribed subsets of the rationals having a desired supremum.

2.) Take $\boldsymbol{R} := \{\boldsymbol{\alpha} \subseteq \mathbb{Q} \mid \boldsymbol{\alpha} \text{ satisfies (i)-(iii)}\} = \{\text{Dedekind cuts}\}$.

To define the operations, declare $\forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \boldsymbol{R}$:

$$\boldsymbol{\alpha} + \boldsymbol{\beta} := \{\boldsymbol{a} + \boldsymbol{b} \mid \boldsymbol{a} \in \boldsymbol{\alpha} \text{ and } \boldsymbol{b} \in \boldsymbol{\beta}\}$$

$$\boldsymbol{\alpha} * \boldsymbol{\beta} := \mathbb{Q}^{<0} \cup \{\boldsymbol{a} * \boldsymbol{b} \mid (\boldsymbol{a} \geq 0) \in \boldsymbol{\alpha} \text{ and } (\boldsymbol{b} \geq 0) \in \boldsymbol{\beta}\}.$$

$$(\boldsymbol{\alpha})_{+}^{-1} := -\boldsymbol{\alpha} := \{-\boldsymbol{q} \mid \boldsymbol{q} \in \mathbb{Q} \backslash \boldsymbol{\alpha} \text{ and } \boldsymbol{q} \neq \boldsymbol{sup}(\boldsymbol{\alpha})\}$$

$$(\boldsymbol{\alpha})_{*}^{-1} := \boldsymbol{\alpha}^{-1} := \{\frac{\boldsymbol{1}}{\boldsymbol{q}} \mid \boldsymbol{q} \in \mathbb{Q} \backslash \boldsymbol{\alpha} \text{ and } \boldsymbol{q} \neq \boldsymbol{sup}(\boldsymbol{\alpha})\}$$

The special elements are given by $\boldsymbol{0} := (-\infty, \boldsymbol{0}) \cap \mathbb{Q}$ and $\boldsymbol{1} := (-\infty, \boldsymbol{1}) \cap \mathbb{Q}$

The order is given by defining:

$$(\boldsymbol{\alpha} \leq \boldsymbol{\beta}) \quad \leftrightarrow \quad \boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$$

• Prop: The collection $(\boldsymbol{R}, +, *, \boldsymbol{0}, \boldsymbol{1}, \leq)$ forms a *totally ordered field.* [Proof: See below.]

On the next page, we will prove that $\boldsymbol{R}$ is closed under both operations $(+/*)$. The rest of the details of the definition of totally ordered field can be checked [Exercise: See the end of Section 1.2.]

2.1: Proof of Closure for $\alpha + \beta$ and $\alpha * \beta$:

• **Prop:** The set of all Dedekind cuts is closed under $(+/*)$.

Proof:

Recall:

$$\alpha + \beta := \{a + b \mid a \in \alpha \text{ and } b \in \beta\}$$

$$\alpha * \beta := \mathbb{Q}^{<0} \cup \left\{a * b \mid (a \geq 0) \in \alpha \text{ and } (b \geq 0) \in \beta\right\}.$$

and $\alpha \in R$ if:

(i) $\emptyset \neq \alpha \neq \mathbb{Q}$
(ii) $\forall a \in \alpha,\ \forall b \in \mathbb{Q},\ (b < a) \implies (b \in \alpha)$
(iii) $\forall a \in \alpha,\ \exists b \in \alpha,\ a < b < sup(\alpha)$

- -

It is clear that both operations yield subsets of $\mathbb{Q}$ as they are defined in terms of elements of other subsets $\alpha, \beta, \mathbb{Q}^{<0} \subseteq \mathbb{Q}$.

(+i) We need only show proper containment. Since α and β satisfy (i), they are nonempty and so each contain an element say $a_0 \in \alpha$ and $b_0 \in \beta$. This implies $a_0 + b_0 \in \alpha + \beta$ meaning $\alpha + \beta \neq \emptyset$.

Since $\alpha, \beta \neq \mathbb{Q}$ we know $\exists q, r \in \mathbb{Q}$ such that $q \notin \alpha$ and $r \notin \beta$. This suggests $q + r \notin \alpha + \beta$ because it fails to satisfy its set condition. Hence $\alpha + \beta \neq \mathbb{Q}$.

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(+ii) Let $x \in \alpha + \beta$ and $y \in \mathbb{Q}$ be arbitrary such that $y < x$. We know then that $\exists a \in \alpha$ and $b \in \beta$ such that $x = a + b$. This implies $y < a + b$, so in particular $y - b < a$ and using axiom (ii) for α, we get that $y - b \in \alpha$. But this means $y = (y - b) + b \in \alpha + \beta$ and by arbitrariness of x, y in their respective sets, we're done.

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(+iii) Suppose $x = a + b$ for arbitrary $a \in \alpha$ and $b \in \beta$. Using the axiom (iii) in each case, we get $\exists y_1 \in \alpha$ and $\exists y_2 \in \beta$ such that $a < y_1 < sup(\alpha)$ and $b < y_2 < sup(\beta)$. But these two imply $\exists y = y_1 + y_2 \in \alpha + \beta$ such that: $x = a + b < y = y_1 + y_2 < sup(\alpha) + sup(\beta) = sup(\alpha + \beta)$. Arbitrariness of $a, b,$ and hence x finishes (+iii).

We conclude that $\alpha + \beta \in R$. $\square$

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(Continues)

Proof(Continued):
In what follows, let us assume both $\boldsymbol{sup(\alpha)}$ and $\boldsymbol{sup(\beta)}$ are non-negative $(\geq \mathbf{0})$. We'll handle the other cases cleverly later.

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(*i) WTS proper containment for product. Don't work too hard! By definition, $\boldsymbol{\alpha * \beta} \supseteq \mathbb{Q}^{<\mathbf{0}}$. So it must be nonempty.

Since $\boldsymbol{sup(\alpha), sup(\beta)} \geq \mathbf{0}$, elements in the complement of $\boldsymbol{\alpha, \beta}$ are necessarily positive. So taking the postulated elements $\boldsymbol{q} \in \mathbb{Q}\backslash\boldsymbol{\alpha}$ and $\boldsymbol{r} \in \mathbb{Q}\backslash\boldsymbol{\beta}$ via (i) for each, $\boldsymbol{q * r} \geq \mathbf{0}$ so is not in $\mathbb{Q}^{<\mathbf{0}}$ and since $\boldsymbol{q} \notin \boldsymbol{\alpha}$, the other component of the union's set definition is failed as well for $\boldsymbol{q * r}$. Thus $\boldsymbol{q * r} \notin \boldsymbol{\alpha * \beta}$, making $\boldsymbol{\alpha * \beta} \neq \mathbb{Q}$. $\quad\therefore \boldsymbol{\alpha * \beta}$ is properly contained in $\mathbb{Q}$.

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(*ii) Suppose $\boldsymbol{x} \in \boldsymbol{\alpha * \beta}$ and $\boldsymbol{y} \in \mathbb{Q}$ are arbitrary with $\boldsymbol{y < x}$. WTS $\boldsymbol{y} \in \boldsymbol{\alpha * \beta}$.

Case 1: If $\boldsymbol{x} \in \mathbb{Q}^{<\mathbf{0}}$ and $\boldsymbol{y < x}$ then $\boldsymbol{y} \in \mathbb{Q}^{<\mathbf{0}}$ and hence in $\boldsymbol{\alpha * \beta}$.

Case 2: Suppose $\boldsymbol{x = a * b}$ for arbitrary $\boldsymbol{a}(\geq \mathbf{0}), \boldsymbol{b}(> \mathbf{0}) \in \boldsymbol{\alpha, \beta}$ respectively. Then $\boldsymbol{y < a * b}$ which implies by the "positive scaling property", we get that $(\mathbf{1}, \boldsymbol{b}) * (\boldsymbol{y}, \mathbf{1}) < (\mathbf{1}, \boldsymbol{b}) * (\boldsymbol{a * b}, \mathbf{1})$. So $(\boldsymbol{y, b}) < (\boldsymbol{a}, \mathbf{1})$. By (ii) for $\boldsymbol{\alpha}$, we have that $(\boldsymbol{y, b}) \in \boldsymbol{\alpha}$. This implies $(\boldsymbol{y, b}) * (\boldsymbol{b}, \mathbf{1}) \in \boldsymbol{\alpha * \beta}$, but this reduces to $(\boldsymbol{y}, \mathbf{1}) \in \boldsymbol{\alpha * \beta}$ as desired.

Reductions made above just invoked the equivalence class definitions (switched representatives). The above can also be phrased as: $\frac{\boldsymbol{y}}{\boldsymbol{b}} = \frac{\mathbf{1}}{\boldsymbol{b}} * \frac{\boldsymbol{y}}{\mathbf{1}} < \frac{\mathbf{1}}{\boldsymbol{b}} * \frac{\boldsymbol{a*b}}{\mathbf{1}} = \boldsymbol{a}$ *etc.*

To finish case 2, Suppose instead that $\boldsymbol{x = a * b}$ for arbitrary $\boldsymbol{a}(\geq \mathbf{0}) \in \boldsymbol{\alpha}$ but $\boldsymbol{b}(= \mathbf{0}) \in \boldsymbol{\beta}$. Then of course $\boldsymbol{x = a} * \mathbf{0} = \mathbf{0}$ and $\boldsymbol{y} < \mathbf{0}$ says $\boldsymbol{y} \in \mathbb{Q}^{<\mathbf{0}}$ and hence $\boldsymbol{y} \in \boldsymbol{\alpha * \beta}$.

This completes (*ii), which says $\boldsymbol{\alpha * \beta}$ has no lower bound for elements in $\mathbb{Q}$.

- -

(*iii) WTS $\boldsymbol{\alpha * \beta}$ has no largest rational now. Let $\boldsymbol{x} \in \boldsymbol{\alpha * \beta}$ be arbitrary. As before there are cases:

Case 1: Suppose $\boldsymbol{x} \in \mathbb{Q}^{<\mathbf{0}}$. Then since $\mathbb{Q}^{<0}$ satisfies (iii), $\exists \boldsymbol{y} \in \mathbb{Q}^{<\mathbf{0}}$ such that $\boldsymbol{x < y < sup}(\mathbb{Q}^{<\mathbf{0}}) \leq \boldsymbol{sup}(\boldsymbol{\alpha * \beta})$.

Case 2: Suppose $\boldsymbol{x} \in \boldsymbol{P} := (\boldsymbol{\alpha * \beta})\backslash\mathbb{Q}^{<\mathbf{0}}$ and $\boldsymbol{P} \neq \boldsymbol{\emptyset}$ (otherwise we're back in the first case). So there exist $\boldsymbol{a, b}(\geq \mathbf{0}) \in \boldsymbol{\alpha, \beta}$ respectively such that $\boldsymbol{x = a * b}$ and by (iiiα), $\exists \boldsymbol{z} \in \boldsymbol{\alpha}$ such that $\boldsymbol{a \leq z \leq sup(\alpha)}$. By the "positive scaling property" we get:

$$\boldsymbol{a * b < z * b < sup(\alpha) * b}$$

and since $\boldsymbol{b < sup(\beta)}$, we have $\boldsymbol{sup(\alpha)} \geq \mathbf{0}$ together with the positive scaling property again yields: $\boldsymbol{sup(\alpha) * b < sup(\alpha) * sup(\beta)}$. So now by transitivity, we have there exists $\boldsymbol{y := z * b} \in \boldsymbol{\alpha * \beta}$ such that:

$$\boldsymbol{x < y < sup(\alpha) * sup(\beta) =: sup(\alpha * \beta)}. \ \square$$

- -

|<<|

Proof (Continued (2)):

We've proven in the case that $\boldsymbol{sup(\alpha), sup(\beta)(\geq 0)}$, (i)-(iii) hold. That is, under these hypotheses, $\boldsymbol{\alpha * \beta} \in R$. We want to consider the cases where either (or both) of the suprema for $\boldsymbol{\alpha, \beta}$ may be negative.

Recall:

$$\boldsymbol{-\alpha := \{-q \mid q \in \mathbb{Q} \backslash \alpha \text{ and } q \neq sup(\alpha)\} = (-\infty, -sup(\alpha)) \cap \mathbb{Q}}$$

WTS as sets, that:

$$\boldsymbol{\alpha * \beta = (-\alpha) * (-\beta) = -((-\alpha) * \beta) = -(\alpha * (-\beta))}$$

so that each of the cases aforementioned can be deduced from our proof in the positive suprema case.

By the interval notation definitions:

$$\boldsymbol{\alpha * \beta = (-\infty, sup(\alpha) * sup(\beta)) \cap \mathbb{Q}}$$

$$\boldsymbol{= (-\infty, (-1) * sup(\alpha) * (-1) * sup(\beta)) \cap \mathbb{Q}} \qquad [\text{i.e. } \boldsymbol{(-\alpha) * (-\beta)}]$$

$$\boldsymbol{= (-1) * (-\infty, (-1) * sup(\alpha) * sup(\beta)) \cap \mathbb{Q}} \qquad [\text{i.e. } \boldsymbol{-((-\alpha) * \beta)}]$$

$$\boldsymbol{= (-1) * (-\infty, sup(\alpha) * (-1) * sup(\beta)) \cap \mathbb{Q}} \qquad [\text{i.e. } \boldsymbol{-(\alpha * (-\beta))}]$$

with the understanding that in the last two lines, the leftmost product is for Dedekind cuts, the proof is complete. ■

At this point, *we* have proven that $\boldsymbol{(R, +, *, 0, 1, \leq)}$ gives a totally ordered field. Moreover, there is a 1-1 correspondence with the *wild real numbers*, $\mathbb{R}$:

$$[\boldsymbol{r} \in \mathbb{R}] \leftrightarrow [\boldsymbol{(-\infty, r)} \cap \mathbb{Q} \boldsymbol{=: \alpha_r} \in \boldsymbol{R}],$$

so we may write:

$$\boldsymbol{R} \cong \mathbb{R}$$

and use the axiomatization as necessary.

2.2: Regarding Deduction vs. Computation

Deduction:

Suppose we are given two cuts $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}$. Then, due to the "totality" and "anti-symmetry" conditions of '$\leq$', we must have:

$$\big[(\boldsymbol{\alpha} \leq \boldsymbol{\beta}) \vee (\boldsymbol{\beta} \leq \boldsymbol{\alpha})\big] \wedge \big[(\boldsymbol{\alpha} \leq \boldsymbol{\beta} \leq \boldsymbol{\alpha}) \implies (\boldsymbol{\alpha} = \boldsymbol{\beta})\big].$$

which says either comparison exists in the order but not both (unless the cuts are the same). Considering the arbitrary case, we may formally write statements φ such as:

$\forall \boldsymbol{\alpha} \in \boldsymbol{A}, \ \exists \boldsymbol{\beta} \in \boldsymbol{B}, \ \ldots$

$\boldsymbol{\alpha} + \boldsymbol{\beta}, \ -\boldsymbol{\alpha} * (\boldsymbol{\beta})^{-1},$

$\boldsymbol{p} * (\boldsymbol{\alpha})^{\boldsymbol{m}} + \boldsymbol{q} * (\boldsymbol{\beta})^{\boldsymbol{n}},$

$\sum_{i=0}^{N} c_i(\boldsymbol{\alpha})^{i},$

$\boldsymbol{\alpha} = \boldsymbol{\beta}_1 * \ldots * \boldsymbol{\beta}_k,$

$\boldsymbol{\alpha} \equiv \boldsymbol{\beta}(\boldsymbol{mod} < \boldsymbol{Rel} >),$

$\boldsymbol{f}(\boldsymbol{\alpha}) \leq \boldsymbol{g}(\boldsymbol{\alpha}),$

$\frac{d}{dx}\big|_{\alpha} f(x),$

$\int_{\alpha}^{\beta} f(x)dx, \ldots$

These are some so called *finite well-formed formulas (WFF's)* declared from the *sub-language*:

$$\mathscr{L}_{\mathbb{R}} := \Big\{\{\neg, \wedge, \vee, \implies, \forall, \exists\}, \{\cap, \cup, \mathbb{R}\backslash(\cdot), \mathscr{P}(\cdot), \subseteq, \emptyset\}, \{\leq, =, <, \in\},$$
$$\{+, *, (\cdot)_{+}^{-1}, (\cdot)_{*}^{-1}, 0, 1\}, \{c_i\}_{i \in I}, \{a, b, c, A, B, C, \ldots \alpha, \beta, \gamma \ldots, \alpha_i \ldots\},$$
$$\{f, g, \ldots\}, \{\tfrac{d}{dx}\big|_{\alpha}, \textstyle\int_{\alpha}^{\beta} f(x)dx\}, \ldots\Big\} \qquad \textbf{(2.2.}\boldsymbol{L}\textbf{)}$$

This language will be implicitly updated as we go. Notice the form of some of them lead into studying primality, linear combinations, Diophantine looking equations, derivatives and integrals, etc. There is deductive power at our disposal–using the different *models* we have for the reals (Wild and Dedekind), together with a choice of *valuation*:

$$\boldsymbol{\nu} : \boldsymbol{Form}(\mathscr{L}_{\mathbb{R}}) \to \mathscr{P}(\mathbb{R})$$

$$\varphi \mapsto \boldsymbol{\nu}(\varphi) \subseteq \mathbb{R}.$$

where $\boldsymbol{\nu}(\varphi)$ is the set of points in the model where an arbitrary statement φ evaluates to **true**. These are also called *truth functions*. They are defined recursively from truth values of primitive statements $\varphi := \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ and a *complete* subset of connectives $\{\neg, \wedge\}$. For more on building *deductive systems*, see [16]!

Regarding Computation:

(Wild) Real numbers are out there. But they are like looking into the night sky with a telescope. If we want to get a grasp on their attributes, we need to implement one or more apparatuses (all of which will have an error tolerance).

For some examples, mathematicians have discovered ways to represent the number $\boldsymbol{\alpha} \in \mathbb{R}$ using: *itineraries* '***LRLRRRLRLRLRL...***' based on its location in each sub-sub-interval of $(\boldsymbol{floor(\alpha), ceil(\alpha)})$ [See [1]]; By *binary* '**01010110100101...**'; Or, by some other *decimal expansions* '**1345.01004059...**' which have their algebraic counterparts as series expansions in powers of a base:

$$\sum_{i=0}^{\infty} c_i(base)^i$$

As in the telescope simile, we can increase magnification but only to a certain extent. This extent is, in Numerical Analysis, referred to as ***truncation error*** or ***roundoff error*** [2]. For example:

$$\mathbf{0.01234|56789...} \ =: \ \mathbf{0.01234} \text{ vs. } \mathbf{0.01235}$$

This error propagates/amalgamates through operations including finitely represented numbers.

Operations in the practical realm consist of different algorithms using the strings of integers to calculate. These are much different from our Dedekind operations $(+/*)$, but are more commonly used.

For example, working with binary expansions of *integers* $\mathbb{Z}$, one can calculate the product:

$$z_1 * z_2 = (a_1 * 2^{k_1} + b_1)(a_2 * 2^{k_2} + b_2) = a_1 a_2 * 2^{k_1+k_2} + a_1 b_2 * 2^{k_1} + b_1 a_2 * 2^{k_2} + b_1 b_2.$$

This can be adapted for (finitely represented) real computation by pre- and post- multiplying/factoring. For example:

$$\mathbf{10.1 * 3.14} \to \frac{\mathbf{101 * 314}}{\mathbf{1000}} = \frac{\text{result}}{\mathbf{1000}} = \text{re.sult}$$

Interplay:

So far, we have subsets of the rationals that satisfy a few conditions and have suprema corresponding to the real number we wish to observe and whose decimal representation can be given (up to an error threshold). This can be summarized roughly by:

$$sup(\alpha) := r \approx \sum_i c_i(base)^i$$

Perhaps the most important use of the Dedekind formulation is to assert the *completeness property*. See the Results Entry 3 for more.

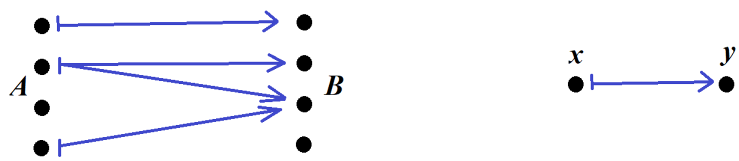
3. Functions and Graphs (esp. Real)

• Def Recall that the *binary cartesian product* of two sets is just:

$$\boldsymbol{A \times B := \{(a,b) \mid a \in A,\, b \in B\}}.$$

• Def If we take two arbitrary sets $\boldsymbol{A, B}$ and assign a collection of **arrows**, $\mapsto$, for pairs of points $\boldsymbol{a, b \in A, B}$, we get a **directed graph**. If we then **label** or **weight** each point, we obtain a **directed, weighted graph**.

Examples:



• Def Notice that existence of an arrow between two points defines a *relation* $\boldsymbol{f}$:

$$[(\boldsymbol{a,b}) \in \boldsymbol{f}] \qquad \leftrightarrow \qquad [\exists\ `\mapsto\text{'} \in \boldsymbol{Arrows(A,B)} \text{ such that } \boldsymbol{a} \mapsto \boldsymbol{b}].$$

Define the **graph of the relation f** to be the subset:

$$\boldsymbol{\Gamma(f) := \{(a,b) \in A \times B \mid (a,b) \in f\}}$$

• Def If $\boldsymbol{\forall a \in A,\ \exists! b \in B,\ (a,b) \in f}$, then we call $\boldsymbol{f}$ a **well-defined relation** or **function**. In this event, we say the **image of** $\boldsymbol{a}$ is $\boldsymbol{b}$ and write either $\boldsymbol{f(a) = b}$ or $\boldsymbol{f : a \mapsto b}$.

We extend this notation to $\boldsymbol{f(A) \subseteq B}$ or $\boldsymbol{f : A \to B}$ at the set level.

Domain, **image of f**, and **codomain of** $\boldsymbol{f}$ are listed respectively by:

$$\boldsymbol{Dom(f) = A,\ Im(f) = f(A),\ Cod(f) = B}.$$

/<</

• <u>Def</u> With the above in mind, we then say the **graph of a function $f : A \to B$** is the subset:

$$\Gamma(f) := \{(a, b) \in A \times B \mid b = f(a)\}$$

• <u>Def</u> We quickly recall some special set-theoretic types of functions:

Injective (1-1): $\forall x, y \in A,\ [f(x) = f(y)] \implies [x = y].$

Surjective (onto): $\forall y \in B,\ \exists x \in A,\ y = f(x).$

Bijective: Both (1-1) and (onto).

• <u>Def</u> In the presence of an **algebraic structure**, we talk about functions preserving the operations. These functions are then usually called **morphisms** or **homomorphisms**. We also have **isomorphisms** for bijective morphisms. We have morphisms whose domain and codomain are the same, we call these **endo-morphisms** with the special case of bijective endo-morphisms being **automorphisms**. These have occasion to be labeled as sets:

$$Hom(A, B),\ End(A) := Hom(A, A),\ Iso(A, B),\ Aut(A) := Iso(A, A).$$

One particular algebraic structure we've considered morphisms for was $\mathbb{R}$. This structure was deemed a *totally ordered field*. It turns out that preserving all of the structure is to rigid a requirement, for example: $End_{(Fields, \leq)}(\mathbb{R}) = \{Id_{\mathbb{R}}\}$ [Exercise: Prove this! Also, explore what happens when we relax all structural preservation conditions and consider turning on some of them at a time!].

• <u>Def</u> Consider $(\mathbb{R}, \leq)$ or any other ordered field (or just poset). We say a function on this field is **increasing** if it preserves the order structure over its domain:

$$\forall x, y \in dom(f),\ [x \leq y] \implies [f(x) \leq f(y)].$$

There is a distinction when the ordering of image points above is strict. In this event, we say **strictly-increasing** or **non-decreasing**. We can then define **decreasing** functions as not strictly-increasing. And **strictly-decreasing** as not increasing.

There are more types of functions we will come to know in the sequel...

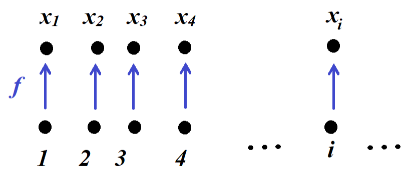
4. Sequences, Convergence, and Limits

We now discuss some "**countable**" or "**discreet**" functions called *sequences*.

• <u>Def</u> Given a set $\boldsymbol{X}$, a **sequence in X** (denoted by $\{\boldsymbol{x_i}\}_{i\in\mathbb{N}}$), is just a function:

$$\boldsymbol{f} : \mathbb{N} \to \boldsymbol{X}$$
$$\boldsymbol{i} \mapsto \boldsymbol{f(i)} =: \boldsymbol{x_i}.$$

<u>Visually:</u>



Sequences are important to us because we can use them to analyze other functions in general. To analyze functions $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ using sequences $\boldsymbol{f} : \mathbb{N} \to \boldsymbol{X}$, we need to define the following:

• <u>Def</u> Let $\boldsymbol{f_1} : \boldsymbol{A} \to \boldsymbol{B}$ and $\boldsymbol{f_2} : \boldsymbol{B} \to \boldsymbol{C}$ be two functions such that either:
(I) $\boldsymbol{Im(f_1)} \subseteq \boldsymbol{Dom(f_2)}$,

(II) $\emptyset \neq (\boldsymbol{Im(f_1)} \cap \boldsymbol{Dom(f_2)}) \neq \boldsymbol{Im(f_1)}$, or

(III) $\boldsymbol{Im(f_1)} \cap \boldsymbol{Dom(f_2)} = \emptyset$ holds.

Then define the **composite function** in each respective case by:

$$\boldsymbol{f_2 \circ f_1} : \boldsymbol{Dom(f_1)} \subseteq \boldsymbol{A} \to \boldsymbol{C}$$
$$\boldsymbol{a} \mapsto \boldsymbol{f_2(f_1(a))}$$

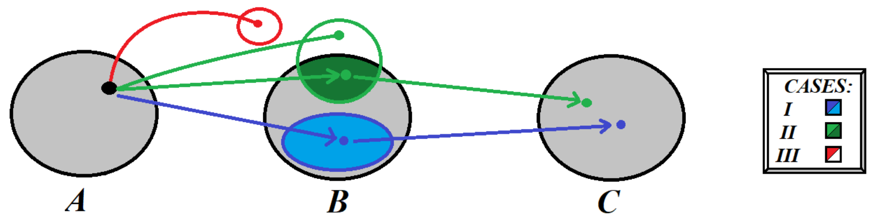
OR

$$\boldsymbol{f_2 \circ f_1} : \boldsymbol{f_1^{-1}}(\boldsymbol{Im(f_1)} \cap \boldsymbol{Dom(f_2)}) \subseteq \boldsymbol{Dom(f_1)} \subseteq \boldsymbol{A} \to \boldsymbol{C}$$
$$\boldsymbol{a} \mapsto \boldsymbol{f_2(f_1(a))}$$

OR

$$\boldsymbol{f_2 \circ f_1} : \emptyset \subseteq \boldsymbol{A} \to \boldsymbol{C}$$
$$\text{null} \mapsto \text{null}$$

Visual for Composition:



• Def Now, suppose we take a sequence $\boldsymbol{f} : \mathbb{N} \to \boldsymbol{X}$ and (without loss of generality–due to the ability to restrict) a function $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ satisfying only (I) above. Then, the composite function forms another sequence $\{\boldsymbol{y_i}\}_{i \in \mathbb{N}}$, where $\forall \boldsymbol{i} \in \mathbb{N},\ \boldsymbol{y_i} := \boldsymbol{g}(\boldsymbol{x_i}) := \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{i}))$. In other words, $\boldsymbol{g} \circ \boldsymbol{f} : \mathbb{N} \to \boldsymbol{Y}$. We then call $\boldsymbol{g} \circ \boldsymbol{f}$ the **image sequence in $\boldsymbol{Y}$**.

Sequences by definition can be all over the place (such as random decimal expansions), but it is the ones we choose to define by formula that make the analysis worthwhile.

$$\text{Example:} \qquad \boldsymbol{x_n} := \mathbf{1} - \frac{\mathbf{1}}{\boldsymbol{n}}$$

Usually we want the values of the sequence to aim in a particular direction, towards points of interest like *singularities* or *extrema*. Most conclusions we make are based on the *limits* of these image sequences.

Suppose we have a *metric* $\boldsymbol{d} : \boldsymbol{X} \times \boldsymbol{X} \to \mathbb{R}^{\geq 0}$ that measures the "**distance**" between points in our sets. Then we can talk about whether or not the values of a sequence approach another value of interest by saying their separation distance approaches zero.

• Def Given a sequence $\{\boldsymbol{x_i}\}_{i \in \mathbb{N}}$ in a *metric space* $(\boldsymbol{X}, \boldsymbol{d})$, we say the sequence **converges** to the **limit point**, $\boldsymbol{x} \in \boldsymbol{X}$, if:

$$\forall \varepsilon > \mathbf{0}, \exists \boldsymbol{N} \in \mathbb{N}, \forall \boldsymbol{n} > \boldsymbol{N},\ \boldsymbol{d}(\boldsymbol{x_n}, \boldsymbol{x}) < \varepsilon.$$

In the event that there exists such a point in the space satisfying this statement, we write:

$$\boldsymbol{x_n} \to \boldsymbol{x}$$

as a shorthand.

Note: Above, we have $\boldsymbol{N} \in \mathbb{N}$. People solve for $\boldsymbol{N} \in \mathbb{R}$ in practice, but technically we should use the ceiling function to get the result in $\mathbb{N}$.

[Exercise: Our example sequence above has limit point at $\boldsymbol{x} = \mathbf{1}$. Prove this using the definition in blue above, together with the metric on the reals given by $\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) := |\boldsymbol{x} - \boldsymbol{y}|$. Note that if $\boldsymbol{X} := (\mathbf{0}, \mathbf{1})$, the limit is not contained in the space... so we would say the limit does not exist in $\boldsymbol{X}$.]

Finally,

• Def Suppose we are given a function between metric spaces $\boldsymbol{f} : (\boldsymbol{X}, \boldsymbol{d_X}) \to (\boldsymbol{Y}, \boldsymbol{d_Y})$, and an arbitrary sequence converging to a point $\boldsymbol{x} \in \boldsymbol{X}$, say $\boldsymbol{x_n} \to \boldsymbol{x}$ with respect to $\boldsymbol{d_X}$.

Then, if $\exists \boldsymbol{y} \in \boldsymbol{Y}$ such that $\boldsymbol{f(x_n)} \to \boldsymbol{y}$ (that is, the *image sequence* converges to $\boldsymbol{y}$ with respect to $\boldsymbol{d_Y}$), we call $\boldsymbol{y}$ the **limit of $\boldsymbol{f}$ as $\{\boldsymbol{x_n}\}_{n\in\mathbb{N}}$ approaches $\boldsymbol{x}$**. This is also denoted:

$$\boldsymbol{y} = \lim_{\boldsymbol{x_n}\to\boldsymbol{x}} \boldsymbol{f(x_n)}.$$

WARNING: Depending on which sequence we choose to approach the value $\boldsymbol{x}$ with, we may get a different limit point $\boldsymbol{y}$. Consider a step function *in $\boldsymbol{X} = \boldsymbol{Y} = \mathbb{R}$ and two sequences approaching from either side of a step–for such an example of a function having multiple limits at a point.*

This a good place to pick up the discussion in our next main topic. Continuity \>>\.

The remainder of this section goes on to discuss some subtleties of sequences and sub-sequences.

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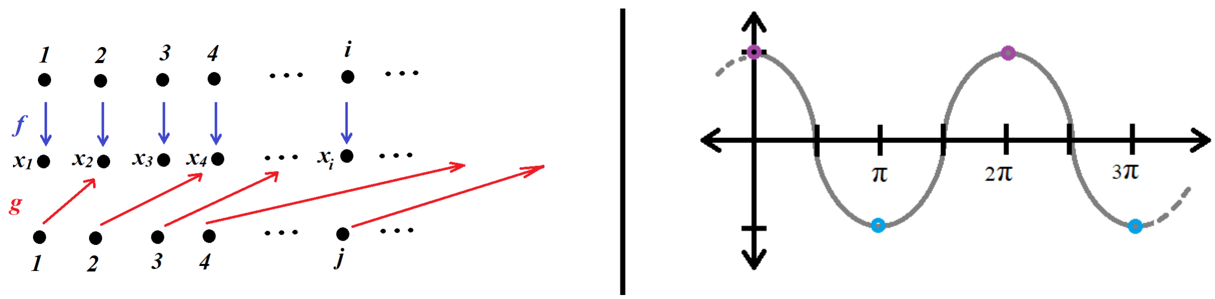
4.1 Subsequences, Lim-Inf, and Lim-Sup



• Def (p.63-64 [15])
Suppose $\{y_j\}_{j\in\mathbb{N}} \subseteq \{x_i\}_{i\in\mathbb{N}}$. Then we know:

$$\forall j \in \mathbb{N}, \exists i \in \mathbb{N}, \text{ such that } y_j = x_i.$$

Now, for any two elements $y_{j_1}, y_{j_2} \in \{y_j\}_{j\in\mathbb{N}}$, denote their correspondents as $x_{i_1}, x_{i_2} \in \{x_i\}_{i\in\mathbb{N}}$. If it is the case that:

$$\forall j_1, j_2 \in \mathbb{N}, \ (j_1 \leq j_2) \implies (i_1 \leq i_2),$$

then we say **$\{y_j\}_{j\in\mathbb{N}}$ is a sub-sequence of $\{x_i\}_{i\in\mathbb{N}}$**.

The last requirement ensures that the subset of elements maintains the order of the original [see the left figure]. This is a desirable property because we want the behavior of the sub-sequence to indicate that of its super. Of course, the super-sequences we wish to analyze usually come from the images of functions.

Consider the simple example motivated by $f(x) = Cos(x)$ [right image above]. If we take the sequence defined by $f_i := f((i-1) * \frac{\pi}{2})$, then there are two subsequences of interest. Namely: $\{1, 1, 1, 1, ...\}$ and $\{-1, -1, -1, -1, ...\}$. It is easy to see that $\{f_i\}_{i\in\mathbb{N}}$ does not have a limit (it oscillates). But the two subsequences do. We wish to capture the limiting behavior of oscillatory sequences in our language. This is done via *lim-inf's* and *lim-sup's*.

• Def For a given sequence $\{x_i\}_{i\in\mathbb{N}}$, define for each index, the subset:

$$T_i := \{x_j \mid j \geq i\},$$

i.e. the *tail of the sequence truncated at* x_i. Then define the following:

$$\{\sup(T_i)\}_{i\in\mathbb{N}} \qquad \text{and} \qquad \{\inf(T_i)\}_{i\in\mathbb{N}}.$$

The limits of these new sequences, when they exist, are respectively referred to as the **limit supremum** and **limit infimum**, denoted

$$\limsup\{x_i\}_{i\in\mathbb{N}} \qquad \text{and} \qquad \liminf\{x_i\}_{i\in\mathbb{N}}.$$

4.2 Cauchy vs. Convergent Sequences; Completeness

The main reference for this subsection is (Section 1.2, pg.9+ of [10]).

Recall earlier in the main section, we defined convergence of a sequence in a metric space:

- -

• Def Given a sequence $\{x_i\}_{i\in\mathbb{N}}$ in a *metric space* (X, d), we say the sequence **converges** to the **limit point**, $x \in X$, if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N,\ d(x_n, x) < \varepsilon.$$

- -

This statement was made in terms of the data: $(X, d, \{x_i\}_{i\in\mathbb{N}}, x)$, where x is the *limit* of the sequence. Say we want to characterize the notion of convergence without reference to the limit. That is to say, define convergence in terms of only the data $(X, d, \{x_i\}_{i\in\mathbb{N}})$.

The idea for convergence is that *the tail of the sequence is contained in a metric ball of radius epsilon and as we increase the index of truncation for the tail, the radius necessary to constrain the tail goes to zero.* This is captured in the following way:

• Def Given a sequence $\{x_i\}_{i\in\mathbb{N}}$ in a *metric space* (X, d), we say the sequence is a **Cauchy sequence** if:

$$\forall \varepsilon, \exists N \in \mathbb{N}, \forall m, n > N, \qquad d(x_m, x_n) < \varepsilon$$

that is, any two points in the N_ε-tail have distance bounded by ε.

Now we have two statements:

Convergent vs. Cauchy

and we want to compare them logically. In studying this, one finds that ***Convergent*** $\implies$ ***Cauchy*** but not necessarily conversely.

- -

Proofs: ($\Rightarrow$) Suppose $x_n \to x$. Then take an arbitrary $\varepsilon > 0$. We have a truncation index N_ε which contains the tail in $B_\varepsilon(x)$. Now, consider the further truncation index $N_{\varepsilon/2} > N_\varepsilon$ containing the "sub-tail" in $B_{\varepsilon/2}(x)$. Then clearly for all $m, n > N_{\varepsilon/2}$, we have by the triangle inequality:

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \varepsilon.$$

Hence the sequence is Cauchy. ■

(Continues)

($\not\Leftarrow$) For the counter-example of the converse, consider the data

$$\big(X := (0,1),\ d(x,y) := |x-y|,\ \{x_n := 1 - 1/n \mid n \in \mathbb{N}\}\big)$$

Given an $\varepsilon > 0$ letting $N := 2/\varepsilon$, we see that if $m, n > N = 2/\varepsilon$ then $\frac{1}{m}, \frac{1}{n} < \frac{\varepsilon}{2}$. So that $d(x_m, x_n) := |1 - \frac{1}{m} - (1 - \frac{1}{n})| = |\frac{1}{n} - \frac{1}{m}| < |\frac{1}{n} + \frac{1}{m}| \le |\frac{1}{n}| + |\frac{1}{m}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ as desired for the sequence to be Cauchy.

But, this sequence has limit $x = 1 \notin X$ so it doesn't converge. [x]

• Def (p.9 [10]) A metric space is **complete** if every Cauchy sequence converges. That is, if we have the dual implication:

Convergent $\Leftrightarrow$ Cauchy.

Notes: The counter example above exploited a case where a sequence would have converged, had the space contained the limit point. Definitions (such as the derivative) that are dependent on the existence of limit points will generally have the hypothesis of *completeness* of the underlying space, as we will see.

Disproving completeness requires one counter-example for a particular space. Proving completeness is more difficult of course and requires further study in general! We assume moving forward that the spaces $\mathbb{R}^n$ are complete with respect to at least the standard metric. See Results Entry 3 for proof.

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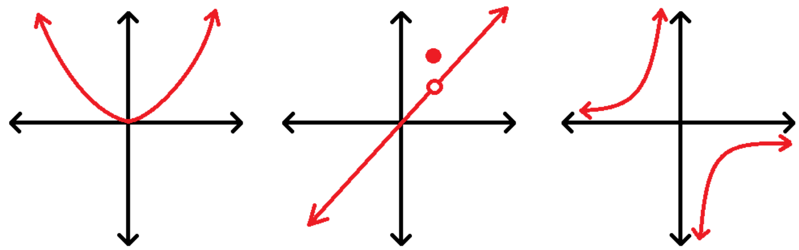
5. Continuity, Derivatives, and Classes of Functions

Recall we had previously defined the limit for a function's image sequence, relative to a choice of sequence in its domain. Furthermore, we noted that this limit may not always be unique.

Now, consider the following examples for real functions:



The left graph is continuous, the right two are not. In the right-most example, there are two limits for the function at zero that disagree ($\pm\infty$). In the middle example, the two limits agree, but the function value doesn't agree with the limits. These two situations characterize continuity.

• Def Given a function between metric spaces $\boldsymbol{f : (X, d_X) \to (Y, d_Y)}$, we say $\boldsymbol{f}$ is **continuous at $\boldsymbol{x_0}$** if its limit value is well-defined (is independent of the domain sequence chosen) and equals the function value at that point. In such a case, we denote:

$$\boldsymbol{f(x_0) = \lim_{x \to x_0} f(x)}$$

The function is said to be **continuous** if it is so at every point in its domain.

Note: The text under "lim" does not specify a named sequence converging to $\boldsymbol{x_0}$, but rather a gives a variable $\boldsymbol{x}$ to be replaced by any desired sequence. The notations we have used differ from [15] a bit.

The definition we have used above is great theoretically, but in practice it is insufficient for proving functions are continuous. The reason being, there are (in the reals) infinitely many sequences available to prove the limits agree over. To reconcile this, mathematicians have come up with another definition that uses only the function and the metric:

• Def $\boldsymbol{f : (X, d_X) \to (Y, d_Y)}$ is continuous at $\boldsymbol{x = x_0}$ if:

$$\boldsymbol{\forall \varepsilon > 0, \exists \delta_\varepsilon > 0, \forall x \in X, \qquad (d_X(x, x_0) < \delta_\varepsilon) \implies (d_Y(f(x), f(x_0)) < \varepsilon)}$$

which says given an epsilon there exists delta such that: $\boldsymbol{x \in B_{\delta_\varepsilon}(x_0)}$ implies $\boldsymbol{f(x) \in B_\varepsilon(f(x_0))}$. Where we use the notation $\boldsymbol{B_r(p) := \{q \mid d(p, q) < r\}}$. A.k.a. the "open d-metric ball of radius r centered at p".

• Prop: The two definitions agree. [Exercise: See Thm 17.2, p.116 [15] for proof!] ■

5.1 Equivalence Proof for The Definition of Continuity:

There are two more alternatives for the definition of continuity:

(1) f is continuous if it ***preserves limits of sequences*** and
(2) if f pulls back *open sets* to *open sets*.

Clearly the last one is topological, this is used in the absence of a metric and we'll come back to it. The first one is an easy consequence of the last two blue statements we've seen above.

Claim: f is continuous *iff* it preserves limits of sequences.

Proof:
($\Rightarrow$) Suppose $f:(X,d_X)\to(Y,d_Y)$ is continuous at an arbitrary point x_0. Then, we write the metric definition:

$$\forall\varepsilon,\exists\delta_\varepsilon,\text{ such that }\forall x\in B_{\delta_\varepsilon}(x_0),\text{ we have }f(x)\in B_\varepsilon(f(x_0)).$$

Now, suppose $x_n\to x_0$. Applying the definition of convergence with parameter δ_ε, we have $\exists N_{\delta_\varepsilon}\in\mathbb{N}$ such that $\forall n>N_{\delta_\varepsilon}$, $x_n\in B_{\delta_\varepsilon}(x_0)$. Our previous assertion says then that $f(x_n)\in B_\varepsilon(f(x_0))$. Restating, we have shown:

$$\forall\varepsilon,\exists N_{\delta_\varepsilon}\in\mathbb{N},\forall n>N_{\delta_\varepsilon},\ f(x_n)\in B_\varepsilon(f(x_0)).$$

Thus $f(x_n)\to f(x_0)$.

- -

($\Leftarrow$) Suppose f preserves limits of sequences–that is, $(x_n\to x_0)\implies(f(x_n)\to f(x_0))$. WTS f is continuous using the metric definition.

Suppose not. Then:

$$\exists\varepsilon'>0\text{ such that }\forall\delta>0,\exists x_\delta\in B_\delta(x_0)\text{ but }f(x_\delta)\notin B_{\varepsilon'}(f(x_0)).$$

In particular, we may construct a decreasing sequence:

$$\delta_1>\ldots>\delta_n>\delta_{n+1}>\ldots>0$$

for which the above statement holds at each node. This in turn provides a sequence $\{x_{\delta_n}\}_{n\in\mathbb{N}}$ that converges to x_0 (the metric balls have progressively smaller radii and are all centered at x_0). However, $\forall n\in\mathbb{N}$, we have $f(x_{\delta_n})\notin B_{\varepsilon'}(f(x_0))$. So it can't be the case that $f(x_{\delta_n})\to f(x_0)$, which contradicts our hypothesis. ■

Derivatives and Regularity Classes of Functions:

We locked down continuity in the last section for the reals, for metric spaces, and even hinted at the topological definition. Continuing in this general manner leads to the *Fretchet Derivative* [see p.47 [10]]. We will list the original definition for $\mathbb{R}$ and work our way up.

• Def Given a function $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}$ that is *continuous in a neighborhood* of a point $\boldsymbol{x_0} \in \mathbb{R}$, the **derivative of $\boldsymbol{f}$ at $\boldsymbol{x_0}$**, is the "limit of the difference quotient", denoted:

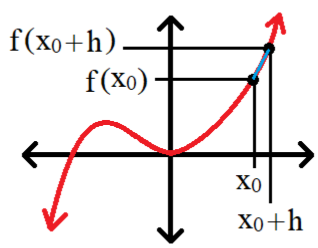
$$f'(x_0) := \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided this limit is well-defined and exists in the codomain.

- -

Notes:

1.) The difference quotient represents the *slope of the secant line* between two points in a neighborhood of $\boldsymbol{x_0}$, which in the limit yields the *slope of the tangent line* to the curve at the point.



2.) We require continuity for this definition to be independent of sequence chosen for $\boldsymbol{h}$ approaching zero (since as we now know, continuous functions preserve limits of sequences).

3.) There are cases where the derivative at a point may not be well defined (consider the absolute value function at zero). The notion of not existing in the codomain as it applies here is when the limits are $\pm\infty$, since we take it to be $\mathbb{R}$. Later this will apply more broadly.

4.) If a function has $\boldsymbol{f'(x_0)}$ existing and well-defined, we say $\boldsymbol{f}$ is **differentiable at $\boldsymbol{x_0}$**. Other notations such as $\frac{d}{dx}f(x)|_{x_0}$ and $\frac{df}{dx}|_{x_0}$ are used to represent the limit.

• Def If a function is differentiable at every point in its domain, we simply say it is *differentiable*. The limit of the difference quotient can then be given a free variable:

$$f'(x) := \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

which defines a function that we call the **derivative of $\boldsymbol{f}$**.

Before moving on, it should be noted that computing derivatives may require limit theorems such as $\lim(f * g) = \lim(f) * \lim(g)$ etc. which we cover in the results section. together with a little algebra.

[Exercise: Try computing the derivative of $f(x) = x^n$ for $(n \geq 0)$.]

Lists of derivatives for libraries of functions can be found everywhere.

• Def Suppose we are given an *open interval*, $(a, b) \subseteq \mathbb{R}$ (we take it to be open to avoid one-sided derivatives). Then the **collection of all continuous functions** on this interval is denoted $C^0\big((a, b)\big)$. Similarly, the **collection of all differentiable functions with continuous derivatives** is denoted $C^1\big((a, b)\big)$. The **collection of all twice-differentiable**... $C^2\big((a, b)\big)$, etc. in general denoted by $C^k\big((a, b)\big)$. The **collection of all infinitely-differentiable functions** is denoted $C^\infty\big((a, b)\big)$. These collections are also called **regularity classes**.

There are still other classes of functions in between these (where the derivatives are not necessarily continuous). As well there are other classes such as **(real) analytic**, denoted $C^\omega\big((a, b)\big)$.

• Def To upgrade derivatives to $f : \mathbb{R}^m \to \mathbb{R}$ one needs to develop **partial derivatives**. These resemble the difference quotient limit above, except the variation is taken in one variable at a time with all others "held constant".

$$\partial_i f(x_1, ..., x_m) := \lim_{h \to 0} \frac{f(x_1, ..., x_i + h, ..., x_m) - f(x_1, ..., x_i, ..., x_m)}{h}$$

We assume continuity over the domain of f for existence and well-definition of these resulting functions $\partial_i f$.

- -

For functions $f : \mathbb{R} \to \mathbb{R}^n$ one simply takes derivatives *component-wise* and gathers the result in to another vector valued function. Similarly for matrix functions $f : \mathbb{R} \to \mathbb{R}^{m \times n}$.

Mappings $f : \mathbb{R}^m \to \mathbb{R}^n$ have a combination of cases happening, we collect the individual component function's partials into column vectors and form a matrix called the **Jacobian Matrix**, which should be denoted by:

$$J_f(x) := \big[\partial_j f^i(x_1, ..., x_m)\big]_{ij}$$

where $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$. Summarizing $J_f(x) \in M_{m \times n}(\mathbb{R}^n)$.

The discussion on regularity varies by the type of function involved for the cases above. In the case of a 1-component function of multiple variables, we have $C^k(\mathbb{R}^n)$ (or over subsets thereof), indicates functions whose ***mixed partial derivative*** exists and is continuous for any order not exceeding k. Multi-index notation gets used. More on this in Section 5.4.

5.2 Normed Vector Spaces; Operators

The following consists of tools for subsection 5.3. The main reference is [Section 7 from [10]].

- Def A **norm** on a (real) vector space is a function:

$$||\cdot|| : V \to \mathbb{R}^{\geq 0}$$
$$v \mapsto ||v||$$

satisfying the following:

1.) $\forall v \in V,\ ||v|| \geq 0$ with equality only when $v = 0$, [Positive-Definiteness]

2.) $\forall \lambda \in \mathbb{R},\ \forall v \in V,\ ||\lambda * v|| = |\lambda| * ||v||$, and [Homogeneity]

3.) $\forall v, w \in V,\quad ||v + w|| \leq ||v|| + ||w||$. [Sub-additivity]

Examples of Normed Spaces:

- $(R^n, ||\cdot||_p)$ - *Real n-component vectors with the* ***p-norm*** *($p \in \mathbb{N}$).*

$$||v||_p := \left[\sum_{i=1}^{n} |v_i|^p\right]^{1/p}$$

The special case where $p = 2$ and $n = 3$ should look familiar.

- -

- $(R^n, ||\cdot||_\infty)$ - *Real n-component vectors with the* ***max-norm****.*

$$||v||_\infty := \max\left\{|v_1|, ..., |v_n|\right\}$$

- -

- $(l^p(\mathbb{R}), ||\cdot||_p)$ - *Sequences with finite (extended)* ***p-norm****.*

$$||v||_p = ||\{v_i\}_{i\in\mathbb{N}}||_p := \left[\sum_{i=1}^{\infty} |v_i|^p\right]^{1/p} < \infty$$

- -

- $(l^\infty(\mathbb{R}), ||\cdot||_\infty)$ - *Bounded sequences with the* ***sup-norm****.*

$$||v||_\infty = ||\{v_i\}_{i\in\mathbb{N}}||_\infty := \sup\left\{|v_i| : i \in \mathbb{N}\right\} < \infty$$

- -

Examples of Normed Spaces (Continued):

Notice that supremum doesn't care about cardinality... we can apply it to arbitrary sets, not just countable sequences. This motivates:

• $(B(X), ||\cdot||_X)$ - Bounded real-valued functions, $f : X \to \mathbb{R}$, with the **sup norm**.

$$||f||_X := \sup_{x\in X} |f(x)| = \sup\{|f(x)| : x \in X\}.$$

>> Special case with $X := \mathbb{N}$ gives $(B(\mathbb{N}), ||\cdot||_{\mathbb{N}}) = (l^\infty, ||\cdot||_\infty)$.

• $(C^0([a,b]), ||\cdot||_p)$ - *Continuous functions over a closed interval, with the (further extended)* ***p-norm***.

$$||f||_p := \left[\int_a^b |f(s)|\right]^{1/p}$$

• Def A **linear operator** between vector spaces satisfies the combined statement referred to as *linearity*:

$$L(\lambda x + y) = \lambda L(x) + L(y).$$

Then the last example we'll list here is:

★ $(B(X,Y), ||\cdot||)$ - *Bounded linear operators,* $L : X \to Y$*, on normed spaces, together with a variant of the sup norm taken over the unit ball.*

$$||L|| := \sup_{x\in \overline{B_1(0)}} ||L(x)||_Y = \sup\left\{||L(x)||_Y : x \in X \text{ and } ||x||_X \le 1\right\}.$$

We say bounded in the sense as before, where we require $||L|| < \infty$.

The fact that any of these satisfy the axioms of a norm is to be proven [Exercise: Check the axioms, some of these are non-trivial. Consult the text [10].] I'll prove the last one for example real quick. To move on with the theory scroll past the proof or click here \>>\.

5.2.1 Sup Norm Proof for B(X,Y)

Claim: The entity $||\cdot||: \boldsymbol{B(X,Y)} \to \mathbb{R}$ defined on two normed spaces by:

$$||\boldsymbol{L}|| := \sup_{\boldsymbol{x}\in\overline{\boldsymbol{B_1(0)}}} ||\boldsymbol{L(x)}||_{\boldsymbol{Y}}$$

satisfies the axioms of a norm.

Proof: We need to show (1) positive-definiteness, (2) homogeneity, and (3) sub-additivity.

1.) Since $\boldsymbol{Y}$ is a normed vector space, $||\boldsymbol{L(x)}||_{\boldsymbol{Y}} \geq \boldsymbol{0}$ with equality iff $\boldsymbol{L(x)} = \boldsymbol{0}$. This implies that $||\boldsymbol{L}|| \geq \boldsymbol{0}$ since it is especially the case that $||\boldsymbol{L(x)}||_{\boldsymbol{Y}} \geq \boldsymbol{0}$ over the unit ball. When $||\boldsymbol{L}|| = \boldsymbol{0}$ it must be the case that $\boldsymbol{L} \equiv \boldsymbol{0}$ since zero is simultaneously the inf and sup for the set of images $||\boldsymbol{L(x)}||_{\boldsymbol{Y}}$ on the unit ball and hence everywhere by linearity. Clearly the converse holds ($\boldsymbol{L} \equiv \boldsymbol{0} \implies ||\boldsymbol{L}|| = \boldsymbol{0}$).

2.) Let $\boldsymbol{\lambda} \in \mathbb{R}$ and $\boldsymbol{L} \in \boldsymbol{B(X,Y)}$ be given. Then we have:

$$||\boldsymbol{\lambda L}|| := \sup_{\boldsymbol{x}\in\overline{\boldsymbol{B_1(0)}}} ||\boldsymbol{\lambda L(x)}||_{\boldsymbol{Y}} = \sup_{\boldsymbol{x}\in\overline{\boldsymbol{B_1(0)}}} |\boldsymbol{\lambda}|\cdot||\boldsymbol{L(x)}||_{\boldsymbol{Y}} = |\boldsymbol{\lambda}|\cdot \sup_{\boldsymbol{x}\in\overline{\boldsymbol{B_1(0)}}} ||\boldsymbol{L(x)}||_{\boldsymbol{Y}} = |\boldsymbol{\lambda}|\cdot||\boldsymbol{L}||$$

since $||\cdot||_{\boldsymbol{Y}}$ satisfies homogeneity. By arbitrariness of $\boldsymbol{\lambda}$ and $\boldsymbol{L}$ the result is proven.

3.) Suppose $\boldsymbol{L}, \boldsymbol{M} \in \boldsymbol{B(X,Y)}$ and let $\boldsymbol{x} \in \overline{\boldsymbol{B_1(0)}}$ be given. Then by sub-additivity of $||\cdot||_{\boldsymbol{Y}}$ and the translation property of the ordering applied twice for the suprema (see Section 1.2), we get:

$$||\boldsymbol{L(x)} + \boldsymbol{M(x)}||_{\boldsymbol{Y}} \leq ||\boldsymbol{L(x)}||_{\boldsymbol{Y}} + ||\boldsymbol{M(x)}||_{\boldsymbol{Y}}$$

$$\leq \sup_{\boldsymbol{y}\in\overline{\boldsymbol{B_1(0)}}} ||\boldsymbol{L(y)}||_{\boldsymbol{Y}} + \sup_{\boldsymbol{y}\in\overline{\boldsymbol{B_1(0)}}} ||\boldsymbol{M(y)}||_{\boldsymbol{Y}} =: ||\boldsymbol{L}|| + ||\boldsymbol{M}||.$$

This being true $\forall \boldsymbol{x} \in \overline{\boldsymbol{B_1(0)}}$ implies it is true in particular for the $\boldsymbol{x_0} \in \overline{\boldsymbol{B_1(0)}}$ such that

$$||\boldsymbol{L(x_0)} + \boldsymbol{M(x_0)}||_{\boldsymbol{Y}} = \sup_{\boldsymbol{y}\in\overline{\boldsymbol{B_1(0)}}} ||\boldsymbol{L(y)} + \boldsymbol{M(y)}||_{\boldsymbol{Y}}.$$

Restating, this gives:

$$||\boldsymbol{L} + \boldsymbol{M}|| = ||\boldsymbol{L(x_0)} + \boldsymbol{M(x_0)}||_{\boldsymbol{Y}} \leq ||\boldsymbol{L}|| + ||\boldsymbol{M}||$$

and by arbitrariness of $\boldsymbol{L}$ and $\boldsymbol{M}$ the result is proven. ■

5.2 Discussion (Continued):

We have seen at this point the definition of normed vector spaces and various examples including that of the bounded linear operators with the sup norm. We'll need to talk about convergence of sequences in norm, but convergence if you recall was defined in terms of metrics. So how do the two compare?

• Prop: Given a normed vector space $(X, ||\cdot||)$ we may define a metric via $d_{induced}(x, y) := ||x - y||$.

Proof:
[Exercise: Easy! Check positive-definiteness, symmetry, and the triangle inequality.] ■

Note: Given a metric $d : X \to \mathbb{R}$, one naively attempts to define a norm via $||x||_{induced} := d(0, x)$, but in attempt to prove *homogeneity* or *sub-additivity*, it doesn't work out from the metric axioms alone! A counter example comes with the *discrete metric* (p.3 [10]):

$$d(x, y) := \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

Take arbitrary $\lambda \neq 0$ with $|\lambda| \neq 1$ and $x \neq 0$. Then $||\lambda x||_d = 1 \neq |\lambda| = |\lambda| \cdot ||x||_d$. In other words, homogeneity fails.

[Exercise: Keeping the induced norm definition, formulate the additional requirements d must satisfy.]

• Def In light of the above, **convergence in norm** means convergence in the norm's induced metric. That is, $x_n \to x$ iff $||x_n - x|| \to 0$.

• Def The **metric topology** or *topology induced by the metric* is simply the closure of the collection of all open balls $B_r(x) := \{y \in X \mid d(x, y) < r\}$ centered at given points $x \in X$ (with respect to axioms (i)-(iii) in Section 1.1 for topological spaces).

The **norm topology** is the norm induced metric topology. That is, the norm induces a metric and we use that metric to give the topology as above.

Not all spaces are *complete*. This is a problem when we need existence of limits. So we make one more definition.

• Def A **Banach space** is a normed vector space that is *complete* in the induced metric.

Without further ado...

5.3 The Fretchet Derivative

This section is based off of (Section I.9, pg.47+ [10], Section 2.1, pgs.15-16 of [18], and [24].)

Recall previously in Section 5, we defined:

Given a function $f : \mathbb{R} \to \mathbb{R}$ *that is* continuous in a neighborhood *of a point* $x_0 \in \mathbb{R}$, *the* ***derivative of f at*** x_0, *is the "limit of the difference quotient", denoted:*

$$f'(x_0) := \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided this limit exists in $\mathbb{R}$*).*

We may rewrite the expression for the derivative as a function f' satisfying:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - h \cdot f'(x_0)}{h} = 0.$$

To upgrade to normed vector spaces, we can incorporate the norms to get us back in $\mathbb{R}$.

★ Def: Suppose we have a function $f : X \to Y$ between *complete* normed, vector spaces. Further, let f be continuous on an open neighborhood U_{x_0} of a point $x_0 \in X$.

If there exists a bounded, linear transformation $L : U_{x_0} \to Y$ such that:

$$\lim_{||h||_X \to 0} \frac{\big|\big|f(x_0 + h) - f(x_0) - ||h||_X \cdot L(x_0)\big|\big|_Y}{||h||_X} = 0$$

then we say the **Fretchet derivative of f at x_0** is the function value $L(x_0)$ and we write:

$$f'(x_0) := L(x_0).$$

Varying the point defines the derivative function, $f'(x)$ so long as the conditions are satisfied.

[Exercise: Prove the Fretchet derivative of $f : \mathbb{R}^m \to \mathbb{R}^n$, when it exists, is the Jacobian matrix $J_f(x) := [\partial_j f^i(x_1, ..., x_m)]_{ij}$. Hint: Consider the defining expression for Fretchet derivative for a single component function and the definition of partial derivative.]

5.4 Partial Differential Operators over $\mathbb{R}^n$

In this subsection, we aim to define notation used in the theory of Partial Differential Equations, starting with *directional derivatives*. We need some preliminary definitions.

• Def

In Section 5 main, we defined partial derivatives for continuous, (real) functions as:

$$\partial_i f(x_1, ..., x_n) = \lim_{h \to 0} \frac{f(x_1, ..., x_i + h, ..., x_n) - f(x_1, ..., x_i, ..., x_n)}{h}.$$

Let us abstract away the function and argue for any fixed $k \in \mathbb{N}$ that the **differential operator**:

$$\partial_i : C^k(\mathbb{R}^n) \to C^{k-1}(\mathbb{R}^n)$$

$$f \mapsto \partial_i(f).$$

So ∂_i reduces the differentiability class by 1. If f is twice differentiable, it's derivative should be only once differentiable, etc. For smooth functions, we get the same class back of course.

- -

The vector space operations, $\{\tilde{\tilde{+}}, \tilde{\tilde{*}}\}$, on $\mathbb{R}$ induce operations, $\{\tilde{+}, \tilde{*}\}$, on $C^k(\mathbb{R}^n)$ (for any k) by doing the algebra in the image of the functions. Concisely, $\mathbb{R}$-linear combinations in $C^k(\mathbb{R}^n)$ are functions defined point-wise via:

$$\big((\lambda \tilde{*} f) \tilde{+} g\big)(x) := \lambda \tilde{\tilde{*}} \big(f(x)\big) \tilde{\tilde{+}} \big(g(x)\big)$$

We can again induce operations, $\{+, *\}$, on the **set of partial differential operators**:

$$PDO_k \supseteq \left\{\partial_i : C^k(\mathbb{R}^n) \to C^{k-1}(\mathbb{R}^n) \;\middle|\; i \in \{1, ..., n\}\right\}$$

using the structure of the image function space $C^{k-1}(\mathbb{R}^n)$. Concisely, $\mathbb{R}$-linear combinations of the partials are defined function-point-wise via:

$$\Big[\big((\lambda * \partial_i) + \partial_j\big)(f)\Big](x) := \Big[\lambda \tilde{*} \big(\partial_i(f)\big) \tilde{+} \big(\partial_j(f)\big)\Big](x)$$

$\therefore$ Together with the zero functions, $0 : \mathbb{R}^n \to \mathbb{R};\ 0(x) :\equiv 0$ and $0 : C^k(\mathbb{R}^n) \to C^{k-1}(\mathbb{R}^n);\ 0(f) :\equiv 0$ respectively, the function spaces for all k and the linear closure of partials (i.e. PDO_k) form vector spaces.

(Next page)

Again, we just defined $\mathbb{R}$-vector space structures on $C^k(\mathbb{R}^n)$ and $\langle \{\partial_i\}_{i\in\{1,\dots,n\}} \cup 0\rangle_{\{+,*\}} = PDO_k$.

★ <u>Def:</u> Define a **directional derivative** as a nonzero element, $X \in PDO_k$. Then we have:

$$X = \sum_{i=1}^{n} \lambda_i * \partial_i$$

for scalars $\lambda_i \in \mathbb{R}$. Moreover, given $f : \mathbb{R}^n \to \mathbb{R}$, we have $X(f) = \sum_{i=1}^{n} \lambda_i * \partial_i(f)$.

• <u>Def</u> Directional derivatives can be used to define what are referred to as **first-order partial differential equations** and there are two types: ***homogenous*** and ***non-homogenous***. Respectively:

$$X(f) = 0 \qquad \text{and} \qquad X(f) = g \qquad (\text{as functions in } C^{k-1}(\mathbb{R}^n)).$$

The name *first-order* comes from the observation that the form of X resembles first-degree (order), multi-variate polynomials: $F(y_1, \dots, y_n) = c_1 \cdot y_1 + \dots + c_n \cdot y_n$. There are specific degree 2 multi-variate poly models that come into play, namely those of ***conic sections***–**parabolic, elliptic, circular, hyperbolic, linear (planar)**. The circular case (a.k.a. ***harmonic***) for example being $y_1^2 + y_2^2$.

If we consider the arbitrary form for a multi-variate polynomial of a given degree, then we can postulate a huge class of **higher-order partial differential equations**. However, we have not yet defined what it means two have say y^2 be replaced by ∂_i^2. This is reconciled by noticing for every $k \geq 0$ we have a collection of partial derivative operators $\partial_i^k : C^k(\mathbb{R}^n) \to C^{k-1}(\mathbb{R}^n)$ and the **composition** between consecutives is defined:

$$[\partial_i^{k-1} \circ \partial_j^k](f) := \partial_i^{k-1}(\partial_j^k(f)).$$

With this full availability, we neglect to write the superscripts each time (instead reserving the space for counting iterations).

• <u>Def</u> On (pgs.2 & 701 [8]), Evans gives **multi-index** notation as a short-hand for the exponents of each partial operator (in order). For example, $\alpha = (1, 2, 3)$ gives a $|\alpha| = 1 + 2 + 3 = 6^{th}$ degree monomial $D^\alpha = \partial_x^1 \partial_y^2 \partial_z^3$. Then, we can write down easily more general **partial differential operators** as:

$$F(\partial_1, \dots, \partial_n) := \sum_{|\alpha| \leq k} \lambda_\alpha \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} = \sum_{|\alpha| \leq k} \lambda_\alpha D^\alpha. \qquad (\bigstar)$$

where all $\lambda_\alpha \in \mathbb{R}$.

(Continues)

Other **classifications** for partial differential operators exist and are dependent on the scalars in each term. That is, are the scalars constants? Do they depend on the underlying spatial coordinates, $\boldsymbol{\lambda_\alpha} = \boldsymbol{\lambda_\alpha(x)}$? etc. We leave these nuances for further study.

• <u>Def</u> **Systems of partial differential operators (and equations)** are relevant to function spaces of type $\boldsymbol{C^k(\mathbb{R}^n, \mathbb{R}^m)}$ as opposed to what we were working with $\boldsymbol{C^k(\mathbb{R}^n) := C^k(\mathbb{R}^n, \mathbb{R})}$. We replace the last form $(\bigstar)$ with a collection of similar forms now also indexed over each component function:

$$\boldsymbol{\left\{[F^j(\partial_1, ..., \partial_n)](f^j(x)) := [\sum_{|\alpha|\leq k} \lambda_{\alpha,j} D^\alpha](f^j(x))\right\}_{j\in\{1,...,m\}} = 0; \qquad = g^j(x),}$$

(as functions in $\boldsymbol{C^0(\mathbb{R}^n)}$)

• <u>Def</u> Lastly, the **operator space**, we've described thus far is summarized algebraically as real function coefficient, multi-variate polynomials in partial variables:

$$\boldsymbol{\big(\mathbb{R}(x)\big)[\partial_1, ..., \partial_n] =: PDO}$$

together with $\boldsymbol{\{+, (\cdot)_+^{-1}, *, \circ\}}$ gives $\boldsymbol{(\mathbb{R} \coprod PDO)}$ a type $\boldsymbol{(2, 1, 2, 2)}$ algebra [Exercise: Axiom check in Preliminaries Section, particularly check compatibility with composition.].

And that concludes the excursion into PDE's for now. Let's start integration theory next.

6. Integration Theory (Riemann, Darboux, and Lebesgue)

In this section, we take as references:

[Ch.6 [15], Ch.5 [21], Ch.5 [2], Ch.3 [18], and Ch.1,2,3 and 6 from [19], [23], and Ch.7 [13]].

That is: Ross, Stewart, Atkinson, Spivak, Stein, Wiki, and Leduc.

Popular function types found in integration theory consist of (but are not limited to):

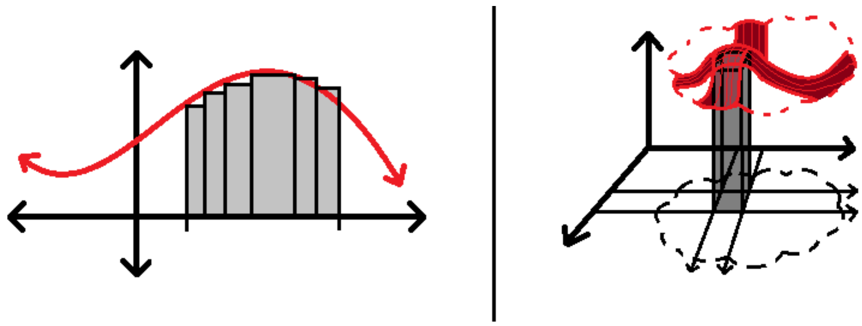
$\boldsymbol{f} : \mathbb{R} \to \mathbb{R}$	1-var Functions
$\boldsymbol{f} : \mathbb{R}^{\boldsymbol{m}} \to \mathbb{R}$	Multi-var Functions
$\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^{\boldsymbol{n}}$	Vector-valued Functions (Space Curves)
$\boldsymbol{f} : \mathbb{R}^{\boldsymbol{m}} \to \mathbb{R}^{\boldsymbol{n}}$	Multi-var, Vector-valued Functions
$\boldsymbol{M} : \mathbb{R} \to \mathbb{R}^{\boldsymbol{m \times n}}$	Matrix Functions
$\boldsymbol{f} : \mathbb{C} \to \mathbb{C}$	Complex Functions
$\boldsymbol{f} : \boldsymbol{\mathcal{M}} \to \mathbb{R}$	Functions on Manifolds
$\boldsymbol{F} : \boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{N}}$	Mappings of Manifolds

Of course, in their contexts they satisfy some additional conditions etc.

We will not touch the complex or manifold theory as these are out of the scope of the project and ultimately are based on real analysis anyways (via $\mathbb{C} = \mathbb{R}^2$ with appropriate operations defining the algebra and for manifolds, through charts $\boldsymbol{\varphi} : \boldsymbol{\mathcal{M}} \to \mathbb{R}^{\boldsymbol{n}}$ we consider the coordinate representations $\hat{\boldsymbol{F}} : \mathbb{R}^{\boldsymbol{m}} \to \mathbb{R}^{\boldsymbol{n}}$) [See [17] for more on tensors and manifold theory]; But, we will get at *multiple integrals*, which correspond to $\boldsymbol{f} : \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}$. The vector valued functions and matrix functions can be had component wise from the base cases that we develop as was the case for derivatives.

At the end of the last section we found that derivatives are also specified for functions $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ on *normed vector spaces*. We will find here that certain spaces called *measure spaces* are appropriate for such abstracted integration. Except, we only integrate functions of type $\boldsymbol{f} : (\boldsymbol{X}, \boldsymbol{\mu}) \to \mathbb{R} \cup \{-\infty, \infty\}$. The connection between measure spaces and metric spaces will be given at the conclusion of this section.

6.1 Riemann and Darboux Integrals



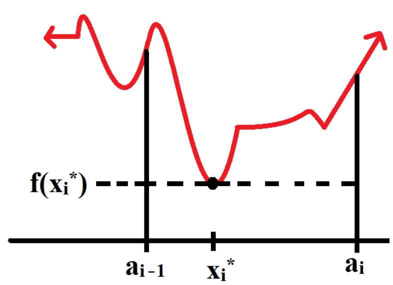
Consider the left figure above. This depicts the process of *approximating area* under positive curves using rectangles (which we know the area of). The better we partition the area to be found, the better the approximation will be in the sum – the limit of this amelioration process being the integral.

This process gets generalized to positive *hyper-surfaces* to approximate *hyper-volume* (analogous to the right figure). The negative and general function value cases follow the positive case.

• Def (p.244 [15]) Given a closed interval $[\boldsymbol{a}, \boldsymbol{b}] \subseteq \mathbb{R}$, a **partition of the interval** is a finite, strictly-ordered subset of points $\boldsymbol{P} \in [\boldsymbol{a}, \boldsymbol{b}]$ for which we want to define as the bottom corners of the rectangles to come. Particularly, we define:

$$\boldsymbol{P := \{a_0, a_1, ..., a_n\}}$$

where $\boldsymbol{a_0 := a,\ a_n := b}$, and $\forall \boldsymbol{i} \in \{\boldsymbol{0, ..., n-1}\},\ \boldsymbol{a_i < a_{i+1}}$.



Considering a given sub-interval for a given partition as in the figure, we have the base length for the rectangle is $\boldsymbol{b_i := |a_i - a_{i-1}|}$ and the height is $\boldsymbol{h_i := |f(x_i^*)|}$ for some point $\boldsymbol{x_i^*} \in [\boldsymbol{a_{i-1}, a_i}]$, which makes the sub-area:

$$\boldsymbol{A_i := h_i * b_i}.$$

Following notational conventions, we write $\boldsymbol{\Delta x_i := b_i = |a_i - a_{i-1}|}$.

• Def (p.250 [15]) For arbitrary choices $x_i^* \in [a_{i-1}, a_i]$ in each **sub-interval** relative to a choice of **partition** P (for some fixed, positive function $f : [a, b] \to \mathbb{R}$), define *a* **Riemann sum** to be:

$$S_P(f) := \sum_{i=1}^{|P|} f(x_i^*) * \Delta x_i = \sum_{i=1}^{|P|} h_i * b_i = \sum_{i=1}^{|P|} A_i = A_P,$$

where $|P|$ denotes cardinality of the set P.

Notes:
1.) As discussed, we next want to figure out a way to define the limiting case where the Riemann sum exactly gives the area under the graph of f. There are several ways for proceeding and each one is based on how we make the choices above.

2.) Just to throw out some examples of choices, one may define uniform sub-interval length $\Delta x_i := \frac{b-a}{n}$ for some integer $n > 0$. Then the x_i^* can be taken to be all **left-endpoints**, **right-endpoints**, or **mid-points** of the sub-intervals. Which one of these is better, depends on the behavior of the function overall. One may choose also the **infimum** and **supremum** for the function values on the sub-intervals. Etc. Techniques for optimizing this calculation are discussed in Numerical Analysis [see **Newton-Cotes** or **Romberg Integration** [2]]. Other topics worth investigating in this field are *Polynomial Interpolation* and *Extrapolation of Derivative Formulas* [2].

★1 Def: As in (p.366 [21]), when the uniform sub-interval length is given by $\Delta x := (b-a)/n$, as well as a uniform choice function for the special points $x_i^* := a + i\Delta x$ (right endpoints), the limit can be characterized in terms of the value for n. In which case *a* **Riemann integral** can be given as:

$$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x := \lim_{n\to\infty} \sum_{i=1}^{n} \frac{f(a + i\frac{(b-a)}{n}) * (b-a)}{n},$$

provided the limit exists.

★2 Def: Described by both [21, 15] respectively on (pg.368, pg.251), with no special choice for the partition, we force the interval lengths to zero in the limit using the so called $mesh(P)$, or the maximum sub-interval length, together with a choice function for the special points we define *a* **Riemann integral** as:

$$\int_a^b f(x)dx := \lim_{mesh(P)\to 0} S_P(f) := \lim_{mesh(P)\to 0} \sum_{i=1}^{|P|} f(x_i^*) * \Delta x_i,$$

provided this limit exists over the set of all partitions of $[a, b]$.

3.) In terms of calculation of examples, ★1 is a good go to. ★2 allows for non-uniform sub-interval length, however explicit details of the partition must be known to formulate the mesh(P) for the limit.

4.) Looking at these definitions, the conditions for existence of $\int_a^b f(x)dx$ is that f be *positive*, *bounded*, and *continuous* over $[a, b]$. This ensures the limit is independent of sequences chosen and that each term in the sum is finite and that we defer the negative and general cases as corollaries.

Next we consider Darboux!

|<<|

Darboux Integrals:

Here, we consider two variants of Riemann sums where the special points are chosen in each sub-interval as to give us the infima and suprema of the function (hinted at in note (2) above).

• Def (pg.243+ [15]) For fixed positive function $f : [a, b] \to \mathbb{R}$ and arbitrary partition P of $[a, b]$. We define the **lower Darboux sum** and **upper Darboux sum** respectively as:

$$L_P(f) := \sum_{i=1}^{|P|} \left(\inf_{x_i^* \in [a_{i-1}, a_i]} f(x_i^*) \right) * \Delta x_i$$

and

$$U_P(f) := \sum_{i=1}^{|P|} \left(\sup_{x_i^* \in [a_{i-1}, a_i]} f(x_i^*) \right) * \Delta x_i$$

- -

In each sub-interval, we took the extreme cases for the height of the rectangles, which implies:

$$L_P(f) \le U_P(f)$$

and the true value for the integral is somewhere in between. Next:

★3 Def: The **lower-** and **upper-Darboux integrals** are defined respectively as:

$$L(f) := \sup_P L_P(f)$$

and

$$U(f) := \inf_P U_P(f).$$

That is, the *lower integral* is the “highest lower sum” and the *upper integral* is the “lowest upper sum”:

$$L_P(f) \le L(f) \qquad \text{and} \qquad U(f) \le U_P(f)$$

When the lower and upper integrals agree, we call the common value the **Darboux integral** and label:

$$\int_a^b f(x)dx = U(f) = L(f).$$

Note:
5.) Aside from positivity being an assumption, *boundedness* of the function f is all that is required to ensure finiteness of the upper and lower integrals and to show the above equality holds [Exercise: See proof of Thm 32.5 [15]]. Continuity is not required!

(Continues)

Multiple Integrals (Riemann/Darboux):

We just described several definitions of integrals, two in particular are listed in blue above for reference. This was for positive, continuous, bounded functions $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}$ and positive, bounded functions of the same type respectively. We extend the function type to $\boldsymbol{f} : \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}$ here with what are called *multiple integrals*, as mentioned before, the other types are had component-wise applying these two base cases. Reference is Ch.5 Spivak now.

• Def **Multiple (Riemann) Integral**:
Suppose we have a *hyper-cube* $\boldsymbol{C} := [\boldsymbol{a_1}, \boldsymbol{b_1}] \times \dots \times [\boldsymbol{a_n}, \boldsymbol{b_n}]$ and a positive, bounded, continuous function $\boldsymbol{f} : \boldsymbol{C} \subseteq \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}$. Define the **multiple Riemann integral** to be the nested integrals:

$$\int_C f(x)dV := \int_{a_n}^{b_n} \left[\int_{a_{n-1}}^{b_{n-1}} \left[\dots \int_{a_2}^{b_2} \left[\int_{a_1}^{b_1} f(x_1, \dots, x_n) dx_1 \right] dx_2 \dots \right] dx_{n-1} \right] dx_n,$$

where to evaluate: we start with the inner-most integral, treat variables $\boldsymbol{x_2, \dots, x_n}$ as constants and integrate with respect to $\boldsymbol{x_1}$, then apply the second integral to the result treating $\boldsymbol{x_3, \dots, x_n}$ as constants etc.

- - -

We interpret the above to be the limiting case of the **multiple Riemann sum**

$$S_P(f) := \sum_{i_n=1}^{|P_n|} \left[\dots \left[\sum_{i_1=1}^{|P_1|} f(x^*_{i_1}, \dots, x^*_{i_n}) \Delta x_{i_1} \right] \dots \right] \Delta x_{i_n} \qquad (\bigstar)$$

relative to a **partition of the hypercube $\boldsymbol{C}$**:

$$P := P_1 \times \dots \times P_n = \left\{ (a_{i_1}, \dots, a_{i_n}) \mid a_{i_j} \in \{0, \dots, |P_j|\} \right\},$$

which as before is strictly-ordered in each component and includes endpoints of the intervals. Moreover there is a **choice function** for the special points in the **sub-hypercubes**:

$$x^*_{i_1 \dots i_n} := (x^*_{i_1}, \dots, x^*_{i_n}) \in [a_{i_1 - 1}, a_{i_1}] \times \dots \times [a_{i_n - 1}, a_{i_n}].$$

• Def **Multiple (Darboux) Integral:**
Considering ($\bigstar$) above. We take the choice function to yield the *infema* and *suprema* of $\boldsymbol{f}$ over the sub-hypercubes $[\boldsymbol{a_{i_1-1}}, \boldsymbol{a_{i_1}}] \times \dots \times [\boldsymbol{a_{i_n-1}}, \boldsymbol{a_{i_n}}]$. Which defines our *lower and upper Darboux sums* respectively. We may then define the *upper and lower Darboux integrals* respectively as the infimum of the upper sums and supremum of the lower sums. When these two agree, we say the function is *Darboux integrable* and we define the **multiple Darboux integral** as the common value.

Final Notes:

6.) In Spivak, (Thm 3-10, pg.58), *Fubini's Theorem* is listed which equates the multiple Darboux integral with embedded single Darboux integrals.

7.) We just developed definitions for multiple integrals over closed hypercubes of positive, bounded, and continuous functions of type $\boldsymbol{f} : \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}$ (where continuity wasn't required in the Darboux case). The negative cases reduce to the positive case with proper use of a negative sign and carefully chosen partitions.

We have neglected to list properties of integrals, but one useful property is that of *additivity of the integral* over disjoint hypercubes. Appealing to 2-dimensional intuition: the area under the curve over two separate closed intervals is just the sum of their separate integrals. This generalized allows us to construct more advanced **domains of integration**.

In particular, Spivak mentions (p.97-100) that we can construct so called *N-chains* which are formal linear combinations of (continuously deformed) hypercubes as the extent to our integration domains. These deformations affect the integrands with appropriate *Jacobian determinants* multiplied to the function – this is best handled with the construction of *differential forms* in Differential Geometry [See [17] for more], but is otherwise known as a *Change of Variable Theorem*.

6.2 Measure Theory and Lesbesgue Integrals

Working from [19] with additional references [23, 13] now.

Summary and Motivation for New Theory:

Recall the "Limit of Riemann Sums" constructions allowed us to calculate multiple integrals of (bounded and continuous) multi-variable functions over hyper-cubes (and hence over $\boldsymbol{N}$-chains by note (7) above). To commit this to symbolism, try:

$$\left(dom(f_{bd,cont}) := \sum_{i=1}^{N} g_i(C_i) \subseteq \mathbb{R}^n\right) \implies \int_{dom(f)} f \qquad \text{is well-defined.}$$

The results we obtain are usually interpreted as some kind of *hyper-volume* between a *hyper-surface* in $\mathbb{R}^{n+1}$ and the $\boldsymbol{x_n = 0}$ plane. For other function types and contexts (such as contour integrals etc.) the interpretation changes.

- -

It turns out that various problems arise when the surrounding theory is developed for the Riemann-based definitions. Four main points are fleshed out in the Introduction of Stein et al. [19]:

(Problems 1 and 2) The space of Riemann integrable functions with the square norm, $(\mathcal{R}, ||\cdot||_2)$, is allegedly not complete as an induced metric space. This means one can find a sequence of functions that are Riemann integrable (bounded/continuous), but the point-wise limiting function is not! We would like these limiting functions to have a notion of integral.

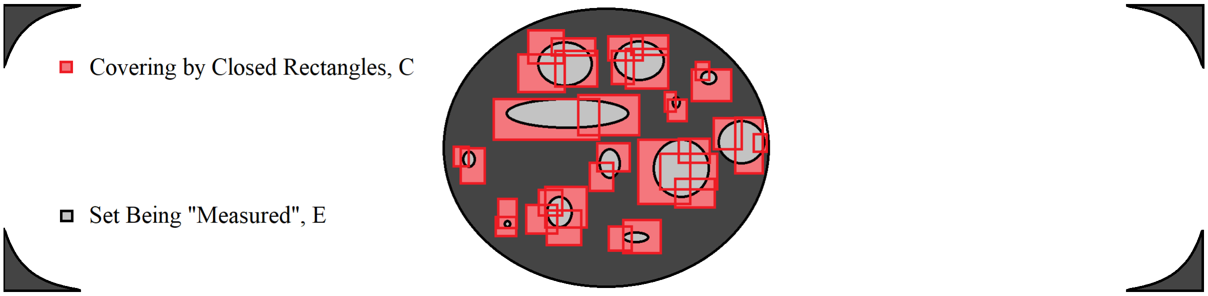
Related to this, the space $(l^2(\mathbb{Z}), ||\cdot||_2)$ of integer sequences with finite square-norm allegedly *is* complete and has the aforementioned space as a *proper subspace* under an identification of each function with their *Fourier coefficients*. We also want a notion of integral for elements in the complement (integer sequences that don't correspond to Riemann integrable functions via Fourier).

(Problems 3 and 4) Regard *curve rectifiability* and *fractional dimension* and (separately) the problems of finding general classes of functions where the *Fundamental Theorem of Calculus Pt.I and II* hold.

Allegedly, Measure Theory and Lebesgue Integration offer solutions to these problems [Exercise: See [19] and follow up!]. We will be concerned mostly with the definitions up through the integral of measurable functions.

(Next Page)

6.2.1 Measure Theory and Integration for Functions of Type $f : \mathbb{R}^n \to \mathbb{R}$



(Motivated by pg.10 [19]) Simply put, we define an *exterior measure* of a set (in n-Dimensional Euclidean space ($\mathbb{R}^n$)) using coverings by hypercubes and summing up all their volumes!

• Def Formally, suppose $E \subseteq \mathbb{R}^n$ and let $\mathcal{C}_E$ denote the **set of all countable coverings of** E **by closed hyper-cubes**. And let such a covering be written as: $C := \left(\bigcup_{i=1}^{\infty} C_i \right) \supseteq E$, where C_i are the closed hypercubes. Then the **exterior measure** is defined as:

$$m_* : \mathscr{P}(\mathbb{R}^n) \to [0, \infty]$$

$$m_*(E) := \inf_{C \in \mathcal{C}_E} \left(\sum_{i=1}^{\infty} |C_i| \right),$$

where $|C_i|$ denotes the volume of C_i and of course $\mathscr{P}$ is the *power set operator*.

Construction Note: (p.13 [19]) This first attempt at defining m_* gives the properties of ***Monotonicity*** and ***Countable Sub-Additivity*** [Exercise: Prove this!], respectively:

$$(E_1 \subseteq E_2) \implies (m_*(E_1) \leq m_*(E_2))$$

and

$$(E = \bigcup_{i=1}^{\infty} E_i) \implies (m_*(E) \leq \sum_{i=1}^{\infty} m_*(E_i)).$$

For a proper notion of *length/area/volume/hyper-volume* using measures, we desire the second property to actually be *equality* when the constituent sets are *disjoint*. However, this does not happen in general [Exercise: Find a c.x.]. To correct this issue we essentially define a class of sets for which the property does hold and we restrict the measure.

★ Def: (p.16 [19]) A subset, $E \subseteq \mathbb{R}^n$, is called **Lebesgue measurable** if:

$$\forall \epsilon > 0, \ \exists O_{open} \supseteq E \qquad \text{such that } m_*(O \backslash E) \leq \epsilon.$$

And for the *set of all Lebesgue measurable sets*, $LMS \subseteq \mathscr{P}(\mathbb{R}^n)$, we define the **Lebesgue measure** as the restriction:

$$m : LMS \to [0, \infty]$$

$$m := m_*\big|_{LMS}.$$

Notes:
Not all sets in Euclidean space are *Lebesgue measurable* (see p.24 [19]), but all sets in Euclidean space are *exterior measurable* (which is to say we can apply $\boldsymbol{m}_*$ to them). Let us take "measurable" to mean "Lebesgue measurable" from now on.

[**Exercise:** Prove Lebesgue measurability is preserved under the set operations: $\left\{ \bigcup_{i=1}^{\infty}, \bigcap_{i=1}^{\infty}, (\cdot \setminus \cdot) \right\}$. See Results Entry 1 for proof.]

[**Exercise:** Prove the Lebesgue measure satisfies Countable Additivity for disjoint collections of measurable sets. See Results Entry 2 for proof.]

Next we move on to define measurable functions.

Measurable Functions:

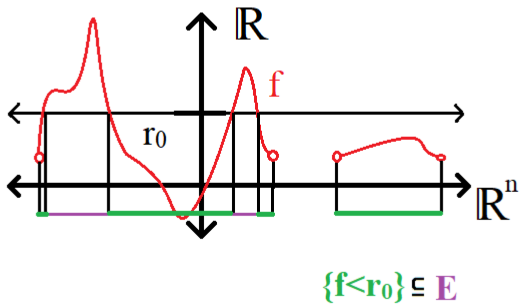
Three different authors have three different definitions for measurable function. One is in terms of pullbacks of *intervals*, the other is in terms of pullbacks of *open sets*, and the last which we'll see later, is in terms of pullbacks of *elements of* $\boldsymbol{\sigma}$*-algebras* (deferred until later).

★ Def:

1.) (p.28[19]) Let $\boldsymbol{f : E \to \mathbb{R}}$ be a function defined on a measurable subset of $\mathbb{R}^n$. Then write:

$$\{f < r_0\} := \{x \in E = dom(f) \mid f(x) < r_0, \text{ for fixed } r_0 \in \mathbb{R}\} \subseteq \mathbb{R}^n,$$

Intuitively:



Then we say $\boldsymbol{f : E_{meas.} \to \mathbb{R}}$ is a **measurable function** if:

$$\forall r_0 \in \mathbb{R}, \ \{f < r_0\} \text{ is measurable.}$$

Equivalently, this says $\boldsymbol{f}$ is measurable if $\forall r_0, \ f^{-1}((-\infty, r_0))$ is measurable.

2.) (p.295 [13]) A function $\boldsymbol{f : \mathbb{R}^n \to \mathbb{R}}$ is said to be a **measurable function** if:

$$\forall O_{open} \subseteq \mathbb{R}, \ f^{-1}(O) \text{ is measurable.}$$

"The pullback of open sets are measurable."

Notes: The equivalence in the definitions above lies in the observation that all open sets in $\mathbb{R}$ can be written uniquely as countable unions of disjoint open intervals [**Thm 1.3, pg.6 [19]**].

The second definition makes it apparent that *continuous functions* are measurable–since open sets in $\mathbb{R}^n$ are trivially Lebesgue measurable. [Exercise: Find a measurable function that is not continuous.]

Approximating Functions (esp. Measurable-) with Simple Functions:

• Def (p.27 [19]) The **characteristic function of a set E** is defined as:

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if} x \notin E \end{cases}$$

• Def (p.27 [19]) A **simple function** is a finite sum of characteristic functions over sets of finite measure. That is:

$$f_{simple}(x) := \sum_{j=1}^{L} \lambda_j \cdot \chi_{E_j}(x),$$

where $\lambda_j \in \mathbb{R}$ and $m(E_j) < \infty$ for all j. Notice that simple functions have $dom(f) = \mathbb{R}^n$.

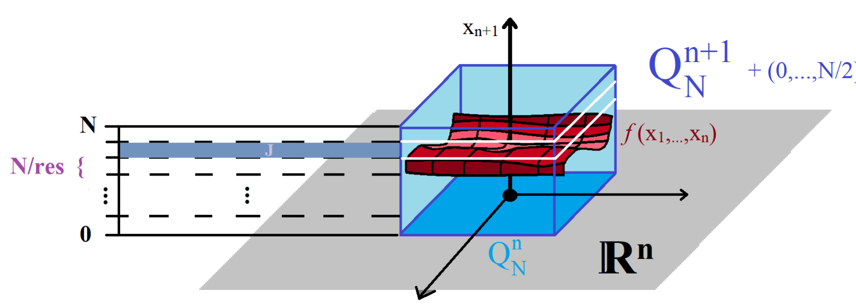
★ Prop: (Thm 4.1, p.31 [19]):

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a positive function, then there exists an increasing sequence of positive, simple functions $\{\varphi_i\}_{i\in\mathbb{N}}$ converging pointwise to f.

- -

This is the last result we need to define the integrals. So, let's go through it in detail.

Proof:
*** My re-interpretation of a re-interpretation of the proof given on (p.31 [19]) by another author [20] ***
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a non-negative function and further suppose that it's domain and range can be contained inside a hypercube of side-length N in $\mathbb{R}^{n+1}$ as the following picture suggests.



Now, uniformly partition the hypercube Q_N^{n+1} along the x_{n+1} coordinate as suggested by the tick marks, up to a *resolution parameter*, $res \in \mathbb{N}$. This way, we define open intervals $\frac{N}{res} * (j, j+1)$, where $j \in \{0, ..., res-1\} \subseteq \mathbb{N}$.

Consider one such arbitrary interval $J := (\frac{N*j}{res}, \frac{N*(j+1)}{res})$.

Focus on only the graph of f that is contained within the *sub-hypercube* $(Q_N^n \times J)$, since we will be defining a new function in terms of this portion of the graph. (Continues)

Proof (Continued):
Define a *pre-approximating function* $F_{N,res,j} : \mathbb{R}^n \to \mathbb{R}$ by:

$$F_{N,res,j}(x) := \begin{cases} \frac{N*j}{res} & \text{if } x \in Q_N^n \text{ and } f(x) \in J \\ 0 & \text{otherwise.} \end{cases}$$

That is to say, we take the lower bound on the interval J to approximate the function value at points that the graph is in the corresponding sub-hypercube. (This is one choice for such an approximating function, one could take the upper bound or anything in between).

If we sum over all the pre-approximating functions for each $j \in \{0, res-1\}$, we get an *approximating function* for f given by:

$$F_{N,res} : \mathbb{R}^n \to \mathbb{R}$$

$$F_{N,res}(x) := \sum_{j=0}^{res-1} F_{N,res,j}(x).$$

Here is the intuitive finale. What does the expression for the $F_{N,res,j}(x)$ resemble if we instead put a 1 in place of $\frac{N*j}{res}$?

These are just specific scaled *characteristic functions*, hence (if the characteristic domains are of finite measure) then the $F_{N,res}(x)$ are all *simple functions*, but this is apparent being contained in a cube. Moreover, they are clearly positive and the higher the resolution, the finer the partition of the range and hence the better the approximating functions.

Appealing to this, for fixed N, define a sequence of simple functions by:

$$\{\widetilde{\varphi}_{N,i}\}_{i\in\mathbb{N}} := \{F_{N,i}(x)\}_{i\in\mathbb{N}}.$$

That is, we index over the *resolution parameter 'i'*.

This completes the construction for the case of the domain and range of a function being contained in a hypercube Q_N^{n+1}. It remains to show that (1) this sequence is increasing in the sense that $\forall x \in \mathbb{R}^n$, if $i_1 \leq i_2$, then $\widetilde{\varphi}_{N,i_1}(x) \leq \widetilde{\varphi}_{N,i_2}(x)$ and (2) that the sequence actually does converge point-wise to f (using an $\epsilon - \delta$ argument) [Exercise: Prove (1) and (2)!].

□

Suppose now that we have an arbitrary, positive function $f : \mathbb{R}^n \to \mathbb{R}$. For any given N, we may reduce to the previous case by defining:

$$f_N(x) := \begin{cases} N & \text{if } x \in Q_N^n \text{ and } f(x) > N \\ f(x) & \text{if } x \in Q_N^n \text{ and } f(x) \leq N \\ 0 & \text{otherwise.} \end{cases}$$

Then as we have shown, to each f_N there corresponds a sequence $\{\widetilde{\varphi}_{N,i}\}_{i\in\mathbb{N}}$ with the desired properties. And since $f_N \to f$ point-wise [Exercise: Prove this!], so does $\{\widetilde{\varphi}_{N,i}\}_{N,i\in\mathbb{N}}$ in the double limit. We may list the indices, (N, i), in an infinite 2D array and define a new counter, k, that zigzags through all of them (similar to cardinality proof for $\mathbb{Q}$), to define $\{\varphi_k\}_{k\in\mathbb{N}}$. Positivity is clear from f being positive, the increasing property should be checked for this new sequence. □

Above, the focus was on the construction of the sequence of simple functions (as the sources of the proof were very unclear about this). Now, up to an error tolerance due to finiteness of for-loops, it is programmable. That being said, the facts regarding term behavior and convergence were deferred as straightforward exercises to keep the length of the proof down!

Finally, the integral construction. See next page.

Lebesgue Integration:

★ Def: (p.49-64 [19]) For a *characteristic function* $\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$ over a measurable set E, we define the integral as:

$$\int_{\mathbb{R}^n} \chi_E(x)dx := m(E).$$

For a *simple, positive, and measurable function*, $f(x) = \sum_{j=1}^{L} \lambda_j \cdot \chi_{E_j}(x)$, taking the *unique* expansion with all distinct coefficients and disjoint characteristic domains (a.k.a. *canonical form*), we define the integral as:

$$\int_{\mathbb{R}^n} f(x)dx := \sum_{j=1}^{L} \lambda_j \cdot m(E_j).$$

If f is a *positive, measurable function*, we have by the previous Proposition that it is the point-wise limit of an increasing sequence of positive, simple functions, $\{\varphi_i(x)\}_{i\in\mathbb{N}}$. We provisionally define the integral as:

$$\int_{\mathbb{R}^n} f(x)dx := \lim_{i\to\infty} \int_{\mathbb{R}^n} \varphi_i(x)dx.$$

*Existence and uniqueness of the limit on the RHS needs to be proven and is the content of (**Lemma 1.2, p.53** [19]).* When f is also *bounded* and *supported on a set of finite measure*, this definition yields a finite result for the integral. When f satisfies all of the above except for boundedness, we must define another step for the *(extended) Lebesgue integral.* This is given by:

$$\int_{\mathbb{R}^n} f(x)dx = \sup\left\{ \int_{\mathbb{R}^n} g(x)dx \;\middle|\; \text{where } g \text{ is bdd., meas., } m(supp(g)) < \infty \text{ and } 0 \le g \le f \right\}.$$

If f is any *measurable function*, we may split it into a sum of a positive and negative function:

$$f(x) := f^+(x) - f^-(x)$$

where $f^+(x) := max(f(x), 0)$ and $f^-(x) := max(-f(x), 0)$. From there, we may define the integral using the previous cases.

Final notes: To define the integral over a measurable subset, $E \subseteq \mathbb{R}^n$ with finite measure, we note in all cases above that:

$$\int_E f(x)dx := \int_{\mathbb{R}^n} f(x) \cdot \chi_E(x)dx.$$

So to perform an integral, find which stage is appropriate for you and if necessary, define the sequence of simple functions by the previous algorithm suggested by the gradient of colors.

Let's move on to the abstract theory!

|<<|

6.2.2 Abstract Measure Theory and Integration for Functions of Type $f : (X, \mu) \to ([-\infty, \infty], m)$

To recap the previous work in sub-subsection 6.2.1: We started by defining an *exterior measure*, $m_* : \mathscr{P}(\mathbb{R}^n) \to [0, \infty]$. Then by restricting m_* to the σ-algebra of *Lebesgue measurable* sets, we were able to give the *Lebesgue measure* $m : LMS \to [0, \infty]$ which satisfied the desired intuitive properties of a volume function for sets in $\mathbb{R}^n$. We then defined *measurable functions*, $f_{meas.}$, as those that preserved measurability of open sets through the pullback. This was important for use of the constructive proof that followed, where we posed measurable functions as limits of sequences of measurable, simple functions. The integral was then constructed in steps, based off these simple functions.

What follows will parallel this story and will be pretty concise.

• Def (p.270 and p.23 [19]) An **algebra** (in the context of Measure Theory) is a set M that is closed under *finite* unions and intersections and also closed under compliments. A **σ-algebra** is a set $\mathcal{M}$ which is closed under *countably infinite* unions and intersections as well as complements. [Exercise: Reconcile these with our Preliminary definition of algebras.]

• Def (p.264 [19]) Given a set X, an **exterior measure** is a function:

$$\mu_* : \mathscr{P}(X) \to [0, \infty]$$

satisfying **monotonicity** and **countable sub-additivity** (as defined previously). A **measure** satisfies **countable additivity**. Technically we have $\mu_*(\emptyset) = 0$ as a property as well (as this is not given in general).

• Def (p.264) A set $E \subseteq X$ is **Caratheodory measurable** or just **measurable** if the following condition holds:

$$\forall A \subseteq X,\ \mu_*(A) = \mu_*(A \cap E) + \mu_*(A \backslash E).$$

This is referred to as the *separation condition* in the context of Measure Theory. [Exercise: Prove that the Lebesgue measurability criterion is equivalent to the above for the case of Euclidean space with the original exterior measure relative to closed cubes.]

By **Thm 1.1 (p.265 [19])**, given an exterior measure on a set X, the collection of Caratheodory measurable subsets of X forms a σ-algebra and the restriction of the exterior measure to this collection yields a measure. Hence we define the **Caratheodory measure**:

$$\mu : CMS \to [0, \infty],$$

where $\mu(E) := \mu_*(E)$ when $E \in CMS$.

(Next page)

• Def (p.263 [19]) A **measure space** in general is a triple $(\boldsymbol{X}, \boldsymbol{\mathcal{M}}, \boldsymbol{\mu})$, where $\boldsymbol{X}$ is a set, $\boldsymbol{\mathcal{M}}$ is a $\boldsymbol{\sigma}$-algebra, and $\boldsymbol{\mu} : \boldsymbol{\mathcal{M}} \to [\mathbf{0}, \infty]$ is a measure.

A measure space is called **σ-finite** if:

$$\exists \{\boldsymbol{X_i}\}_{i \in \mathbb{N}}, \text{ where } \forall \boldsymbol{i},\ \boldsymbol{X_i} \subseteq \boldsymbol{X},\ \boldsymbol{\mu}(\boldsymbol{X_i}) < \infty, \text{ and } \boldsymbol{X} = \bigcup_{i \in \mathbb{N}} \boldsymbol{X_i}.$$

*This is a technical condition used in the proof of the general case for the simple-function-sequence-pointwise-convergence-theorem that lies at the heart of the integral construction. However, the author leaves it as a problem for the reader [**Exercise/Problem**].*

• Def (p.273 [19]) Given a measure space $(\boldsymbol{X}, \boldsymbol{\mathcal{M}}, \boldsymbol{\mu})$, an *extended real-valued* function $\boldsymbol{f} : \boldsymbol{X} \to [-\infty, \infty]$ is said to be a **measurable function** if:

$$\forall \boldsymbol{r_0} \in \mathbb{R},\ \boldsymbol{f}^{-1}([-\infty, \boldsymbol{r_0})) \in \boldsymbol{\mathcal{M}}.$$

• Def [23] More generally, given a function between measure spaces, $\boldsymbol{f} : (\boldsymbol{X}, \boldsymbol{\mathcal{M}}, \boldsymbol{\mu}) \to (\boldsymbol{Y}, \boldsymbol{\mathcal{N}}, \boldsymbol{\nu})$, we say $\boldsymbol{f}$ is **measurable** if:

$$\forall \boldsymbol{E'} \in \boldsymbol{\mathcal{N}},\ \boldsymbol{f}^{-1}(\boldsymbol{E'}) \in \boldsymbol{\mathcal{M}}.$$

However, unless we post-compose with a function into the extended reals, integrals won't be defined. Using norms when they are available helps solve this issue.

• Prop: (p.274 [19]) Suppose $\boldsymbol{f}$ is a non-negative function on a measure space $(\boldsymbol{X}, \boldsymbol{\mathcal{M}}, \boldsymbol{\mu})$. Then there exists an increasing sequence of simple functions $\{\boldsymbol{\varphi_i}\}_{i \in \mathbb{N}}$ that converges point-wise to $\boldsymbol{f}$.

Caratheodory Integrals, $\int_E f d\mu$:

With the above upgraded proposition in place, and with another existence and uniqueness proof, the integral definitions starting from the case of *characteristic functions*, through *simple functions*, and up to general *measurable functions* is identical to what we've done in the previous section.

(Continues)

Final Notes:

1.) As promised, we observe the connection to metric spaces:

Thm 1.2 (p.267 [19]): If μ_* is a *metric exterior measure* on a metric space X, then the *Borel sets* in X are measurable. Hence μ_* restricted to $\mathcal{B}_X$ is a measure [and the integral constructions proceed accordingly for functions whose domain is Borel].

- -

• Def (Motivated by p.23/p.267) Given a metric space (X, d), a **Borel σ-algebra on X**, denoted $\mathcal{B}_X$, is the smallest (w.r.t. inclusion) σ-algebra containing all *open sets*. The elements of this algebra are called **Borel sets**.

We have not technically defined open sets *for metric spaces yet, though we have been using them throughout this text. This will be covered in the next section.*

• Def (p.267) An exterior measure μ_* on a metric space (X, d) is called a **metric exterior measure** if it satisfies the following separation axiom in terms of the metric:

$$\forall A, B \subseteq X, \ \Big(d(A, B) > 0\Big) \implies \Big(\mu_*(A \cup B) = \mu_*(A) + \mu_*(B)\Big),$$

where $d(A, B) := \inf \{d(a, b) \mid a \in A \text{ and } b \in B\}$.

2.) (Motivated by p.30/p.53/p.276 [19])

• Def A property is said to be true **almost everywhere**, abbreviated **(a.e.)**, if the property is true except on a set of measure zero. We may define a relation:

$$f \sim g \Leftrightarrow f \equiv g \qquad \text{(a.e.)}$$

And this of course is an equivalence relation.

Integrable functions that agree almost everywhere have the same integral value (argue starting with simple functions and build up). Thus integrable functions on a measure space can be collected into equivalence classes. The collection of all such equivalence classes $[f]_\sim$ together with the *p-norm* given by:

$$||f||_{L^p(X,\mu)} := \left(\int_X |f(x)|^p dx\right)^{\frac{1}{p}}$$

is called an **L^p space over (X, μ)**. These spaces warrant their own further study.

We've finished the basics of Measure Theory. Now is a good time to reflect, we are leaving the land of derivatives and integrals and entering the realm of pure topology. Be sure to visit the comprehensive exam problems section and Results section where some detailed proofs are or will be located as time goes on.

7. Topics in Topology

Introduction

What was covered:

In Sections 1-6, the study of a vast majority of Real Analysis was covered/introduced. Due to the logical lens applied at the startup, we also covered majority of topics seen in both Metric- and Point/Set-Topology.

In the *object* viewpoint, we saw Set operations $\{\cup, \cap, (\cdot)^c, (\cdot \setminus \cdot \cdot)\}$, sequences $\{x_i\}_{i\in\mathbb{N}}$, limits, convergence $x_n \to L$, completeness (in a caveat #ResultsEntry3), boundedness, products and coproducts of Sets/Metric spaces $A \times B$; $A \amalg B$ (also Results Entry 3), a notion of separated sets (in Lebesgue sense, which is different of course from T_1 etc. but still), a notion of bases and generated sets $S = \langle B \rangle_{w.r.t.\ ops}$ (we saw in an algebraic setting for the construction of σ-algebras at least), and cardinal considerations such as discrete vs. countable vs. uncountable.

From the *morphism* viewpoint (i.e. Functional Analysis as a sub-branch of R.A. and Topology): We described various types of $Hom_{\mathcal{C}}(A, B)$ structures in Section 3, the ones that took the spotlight throughout the paper had additional properties like continuous, differentiable, integrable, measurable, bounded etc.

What was NOT covered:

Somehow the discussion of open and closed was postponed and implicitly used throughout via employing metric balls, neighborhoods, and the dichotomy given by the complement operator (as in the proof of E^c being measurable (Results Entry 1). We can elaborate more on these and other properties: open, closed, bounded, and compact.

Different topological bases satisfy different properties. *This topic leads to the separation properties (like Hausdorff) and also to cardinality of bases considerations like Second-Countable. These are two important axioms that, together with (locally Euclidean), give the axiomatic basis for Manifold Theory / Differential Geometry, so I want to cover them in the topological setting.*

I also want to cover subspaces and relative topology and embeddings and induced topology vs. inherited.

In the discussion of completeness, we focused on spaces that *are* complete. We didn't dwell on spaces that *are not* complete.

>> *This path leads straight to Homotopy and Homology Theories (in Algebraic Topology). I want to cover connectedness arguments since those are used in proofs, however the construction of fundamental groups through equivalence classes of closed loops, fiber and covering spaces, chain complexes etc. is all too much for this paper and I've written it before in other projects (up through to the Categorical level) [see my Category Theory Proper and Monodromy Representation projects on the* ***website****].*

The main topics listed in red above will be minimally reconciled in the sequel. The main reference for this section is (pg.59+ [10]).

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7.1: Basic Properties and Constructions

<u>SubSection Menu:</u>

Open vs. Closed:

Consider the *intervals*:

$$[a, b], [a, b), (a, b], \text{ and } (a, b)$$

as subsets of $\mathbb{R}$. We refer to these as respectively: *closed*, *half-open*, *half-open*, and *open* intervals. These are great primitive examples of the more general counterparts because they show containment (or lack of containment) of the limit points of the subsets (a.k.a. *boundary points*). Furthermore, one can see that sets are not just open or closed. They can be *both open and closed* or *neither*.

• Def We use **closed metric balls** and **open metric balls** for the prototypical sets in $\mathbb{R}^n$ and (X, d) in general:

$$\overline{B_\epsilon(x)} := \{y \mid d(x, y) \leq \epsilon\} \quad \text{and} \quad B_\epsilon(x) := \{y \mid d(x, y) < \epsilon\}.$$

Notice that $(a, b) = B_{\frac{b-a}{2}}(\frac{a+b}{2})$ *etc. For the more exotic sets, one must contrive them with combinations of set operations.*

At the fundamental level, the difference between open and closed metric balls boils down to containment of the set of points on the **boundary "spheres"**:

$$\partial B_\epsilon(x) := \{y \mid d(x, y) = \epsilon\}.$$

We want to identify *boundary points* for subsets in general and distinguish them from *non boundary points*. This is done next using the only things at our disposal (more metric balls).

• Def A point x is said to be **adherent to A** if every epsilon ball has non-empty intersection with A. Define the set:

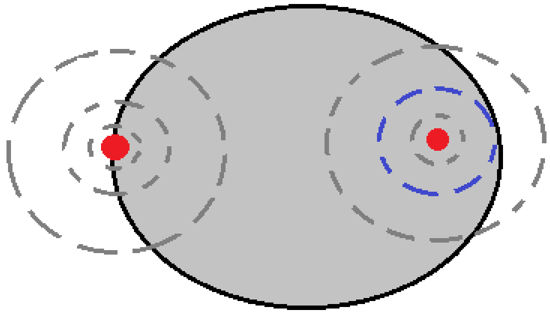
$$adh(A) := \{x \in X \mid \forall \epsilon > 0,\ B_\epsilon(x) \cap A \neq \emptyset\}.$$

A point x is said to be **interior to A** if there exists a threshold radius where $B_\epsilon(x) \subseteq A$. Define the set:

$$int(A) := \{x \in X \mid \exists \epsilon > 0,\ B_\epsilon(x) \subseteq A\}.$$

[Exercise: Show $int(A) \subseteq A \subseteq adh(A)$.]

(Continues)



(Boundary vs. Adherent vs. Interior)

• <u>Def</u> We define the **boundary of A** ($\subseteq (X, d)$) to be the set difference:

$$\partial A := adh(A) - int(A).$$

<u>Notes:</u>
1.) Pure non-containment of the boundary gives us $\partial A \cap A = \emptyset$ which implies by DeMorgan:

$$(adh(A) \cap (int(A))^c)^c \cup A^c = X$$

$$\implies \quad (adh(A))^c \cup int(A) \cup A^c = X$$

Since $A \subseteq adh(A)$, we have $(adh(A))^c \subseteq A^c$ which makes the above reduce to:

$$int(A) \cup A^c = X$$

by combining the outer union components. Converting back now:

$$(int(A))^c \cap A = \emptyset$$

$$\implies \quad A - int(A) = \emptyset$$

Which says finally that $A = int(A)$.

2.) $\partial A \subseteq A$ implies $adh(A) \subseteq A$. Hence $adh(A) = A$ (by the above exercise for the other containment). This prompts:

• <u>Def</u> $A \subseteq (X, d)$ is **open** iff $A = int(A)$. A is **closed** iff $A = adh(A)$.

This discussion readily generalizes.

• Def In a topological space $(X, \mathscr{T})$, we may define the **interior of A** as:

$$int(A) := \{x \in X \mid \exists U \in \mathscr{T} \text{ such that } x \in U \subseteq A\}.$$

The **adherent points of A** can be defined as:

$$adh(A) := \{x \in X \mid \forall U \in \mathscr{T} \text{ such that } x \in U,\ U \cap A \neq \emptyset\}.$$

The **boundary of A** is then:

$$\partial A := adh(A) - int(A).$$

• Def As before, considering containment of the boundary leads to characterizing **open** and **closed** subsets of $(X, \mathscr{T})$ by the respective identities: $int(A) = A$ and $adh(A) = A$.

There are other useful characterizations derivable from these. For example:

• Def/Prop: $A \subseteq (X, \mathscr{T})$ is *closed* iff A^c is open.

[Exercise: Proof: Show $(adh(A) = A) \leftrightarrow (A^c = int(A^c))$.]

(Next page)

Boundedness and Compactness

• Def In a metric space, we have existence of a threshold radius for containment of a subset in a metric ball as the notion of bounded. Formally, $\boldsymbol{A \subseteq (X, d)}$ is **bounded** if:

$$\exists \epsilon > 0 \text{ and } \exists x \in X \text{ such that } A \subseteq B_\epsilon(x).$$

An immediately related notion is:
• Def The **diameter** of a set $\boldsymbol{A \subseteq (X, d)}$ is defined as:

$$diam(A) := sup\{d(x, y) \mid \text{For } x, y \in A\}.$$

Notice that if $diam(A) < \infty$, *then* A *is bounded, since we may take any* $x_0 \in \overline{A}$ *and define* $\epsilon_0 := diam(A)$. *Then* $\forall y \in A$, *we have* $d(x_0, y) \le diam(A) = \epsilon_0$. *Hence* $y \in \overline{B_{\epsilon_0}(x_0)}$. *The containment of* A *follows. We can pick a larger radius to get containment in an open ball.*

• Def In both metric and topological spaces, define $A \subset X$ to be **compact** if for every covering of A by open sets, there exists a *finite subcover*:

$$(A \subseteq \bigcup_{i \in I} S_i \text{ (open)}) \implies (\exists \{i_1, ..., i_n\} \subseteq I, \text{ with } A \subseteq \bigcup_{j=1}^{n} S_{i_j}).$$

There are subtle relationships between the properties: closed, bounded, and compact in metric spaces [Exercise: See (Thm 5.5 (Hiene-Borel), pg.21 [10]) to start.] The subtleties deepen when we try to go to topological spaces. The implications of boundedness and compactness as they relate to functions will be considered all at once with the work in the Seperation Axioms etc. to follow.

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Subspace and Induced Topologies:

• Def (pg.64 [10]) Consider $\boldsymbol{A} \subseteq (\boldsymbol{X}, \mathscr{T}_{\boldsymbol{X}})$. Then if we define:

$$\mathscr{T}_{\boldsymbol{A}} := \{\boldsymbol{U} \cap \boldsymbol{A} \mid \boldsymbol{U} \in \mathscr{T}_{\boldsymbol{X}}\} \subseteq \mathscr{P}(\boldsymbol{A}),$$

$(\boldsymbol{A}, \mathscr{T}_{\boldsymbol{A}})$ is called a **subspace of** $(\boldsymbol{X}, \mathscr{T}_{\boldsymbol{X}})$ with the **relative topology**.
The sets $\boldsymbol{U} \cap \boldsymbol{A}$ are called **relatively open in** $\boldsymbol{A}$ since $\boldsymbol{U} \cap \boldsymbol{A}$ may not be open in $\boldsymbol{X}$.

Notes: $\emptyset, \boldsymbol{A} \in \mathscr{T}_{\boldsymbol{A}}$ *since* $\boldsymbol{U} := \emptyset, \boldsymbol{V} := \boldsymbol{X}$ *are in* $\mathscr{T}_{\boldsymbol{X}}$. *Intersections and Unions are Associative and Commutative and* $\boldsymbol{A} \cap \boldsymbol{A} = \boldsymbol{A} \cup \boldsymbol{A} = \boldsymbol{A}$ *so gathering all the* $\boldsymbol{A}$*'s together gives us closure). So the axioms of a topological space are satisfied.* □

• Def Suppose instead that $(\boldsymbol{A}, \mathscr{T}_{\boldsymbol{A}})$ and $(\boldsymbol{X}, \mathscr{T}_{\boldsymbol{X}})$ are topological spaces in their own right and there exists an injection $\boldsymbol{\iota} : \boldsymbol{A} \hookrightarrow \boldsymbol{X}$, that is $\boldsymbol{A}$ is up to identification a subset of $\boldsymbol{X}$. Then we have an **induced topology** on $\boldsymbol{\iota}(\boldsymbol{A}) \subseteq \boldsymbol{X}$ given by the collection of images:

$$\mathscr{T}_{\boldsymbol{\iota}(\boldsymbol{A})} := \{\boldsymbol{\iota}(\boldsymbol{U}) \mid \boldsymbol{U} \in \mathscr{T}_{\boldsymbol{A}}\}.$$

Notes:
Clearly $\emptyset, \boldsymbol{\iota}(\boldsymbol{A}) \in \mathscr{T}_{\boldsymbol{\iota}(\boldsymbol{A})}$, *since* $\boldsymbol{U} := \emptyset$ *and* $\boldsymbol{V} := \boldsymbol{A}$ *are in* $\mathscr{T}_{\boldsymbol{A}}$ *and* $\boldsymbol{\iota}(\emptyset) = \emptyset$.
It remains to be shown however that:

$$\boldsymbol{\iota}(\boldsymbol{U}) \cup \boldsymbol{\iota}(\boldsymbol{V}) = \boldsymbol{\iota}(\boldsymbol{U} \cup \boldsymbol{V}) \qquad \textit{and} \qquad \boldsymbol{\iota}(\boldsymbol{U}) \cap \boldsymbol{\iota}(\boldsymbol{V}) = \boldsymbol{\iota}(\boldsymbol{U} \cap \boldsymbol{V}),$$

at the atomic level. Then we may use closure of $\boldsymbol{U} \cup \boldsymbol{V}$ *and* $\boldsymbol{U} \cap \boldsymbol{V}$ *in* $\mathscr{T}_{\boldsymbol{A}}$ *to yield* $\boldsymbol{\iota}(\boldsymbol{U} \cup \boldsymbol{V}), \boldsymbol{\iota}(\boldsymbol{U} \cap \boldsymbol{V}) \in \mathscr{T}_{\boldsymbol{\iota}(\boldsymbol{A})}$. *Which after recursion proves* $(\boldsymbol{\iota}(\boldsymbol{A}), \mathscr{T}_{\boldsymbol{\iota}(\boldsymbol{A})})$ *is a topological space contained in* $(\boldsymbol{X}, \mathscr{T}_{\boldsymbol{X}})$. The proof is deferred to Results Entry 4. □

Again, the sets in $\mathscr{T}_{\boldsymbol{\iota}(\boldsymbol{A})}$ are open (relatively), as they may not actually be open in the subspace topology.

[Exercise Explore *coinciding topologies* topic in general and in the cases above for subspace vs. induced topologies.].

Reiterating, we've encountered two different topologies on a subset $\boldsymbol{A} \subseteq (\boldsymbol{X}, \mathscr{T})$: the Relative Topology and the Induced Topology (when we view $\boldsymbol{A} = \boldsymbol{\iota}(\widetilde{\boldsymbol{A}})$ for choice of injection $\boldsymbol{\iota} : \widetilde{\boldsymbol{A}} \hookrightarrow \boldsymbol{X}$). There are more topologies that can be defined on $\boldsymbol{A}$. These are studied in the next section from the *super-space* point of view.

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Generated Topologies
and Structure Axioms (Countability and Separation):

• Def Recall a **topology,** $\mathscr{T}_X$ **on a set** X is just a collection of subsets $\mathscr{T}_X := \{S\}_{\alpha\in A} \subseteq X$ satisfying *algebraic closure axioms* in terms of:

$$\bigcup_{i\in I},\ \bigcap_{i=1}^{N},\ \{\emptyset, X\}.$$

Again: $\mathscr{T}_X \subseteq (\mathscr{P}(X) := \{S \mid S \subseteq X\})$. We can choose any particular collection of subsets $G \subseteq \mathscr{P}(X)$ and generate:

$$\mathscr{T}_G := \langle G \rangle_{\bigcup_{i\in I},\ \bigcap_{i=1}^{N},\ \{\emptyset, X\}} \subseteq \mathscr{P}(X).$$

by recursively including images of the operators.

We call $\mathscr{T}_G$ the **topology generated by** G.

Notes:
1.) Each $\mathscr{T}_G$, depending on intrinsic attributes of the space, will unveil varying levels of geometric information. The extreme cases being of course when $G := \emptyset$, or $G := \mathscr{P}(X)$. If G is empty, we have nothing to compare; $G := \mathscr{P}(X)$, we have potentially too much to compare.

2.) What "local information" can we hope to establish with appropriate choice of topology? Connectedness, path connectedness, maybe more...

• Def Note that in the literature (at least in [10]), authors speak of **bases for topologies**, $\mathcal{B} \subseteq \mathscr{P}(X)$. The elements in $\mathcal{B}$ are called **basic sets** and they generate the topology $\mathscr{T}_{\mathcal{B}}$ *only via union.*

This is opposed to our *generating sets* and *generated topologies* over finite intersections as well as arbitrary unions. This is to say bases are generating sets up to an additional requirement that intersections beith expressible as unions.

Why use bases over generating sets theoretically? Are they just practically easier to write down? The following concepts, though applying to generating sets, are stated in terms of bases $\mathcal{B}$ to match the discourse.

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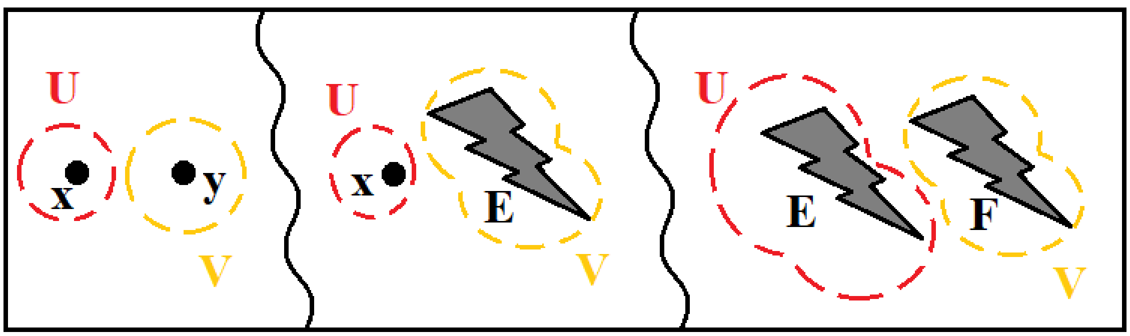
• Def (p.71 [10]) Given a space with generated topology by a basis, $\big(\boldsymbol{X}, \mathscr{T}_{\boldsymbol{X}} := \langle \boldsymbol{\mathcal{B}} \rangle\big)$, if the basis has *countable* cardinality, then we say $\boldsymbol{X}$ is **Second Countable**.

Notes:
Second countable spaces ensure that coverings (esp. open) need at most be countable unions, since every subset $\boldsymbol{A} \subseteq \boldsymbol{X}$ can be written as a union of elements of $\boldsymbol{\mathcal{B}}$, which is itself countable. This is important for things requiring sequential arguments and for the connection to compactness. In manifold theory, requiring Second countability gives the *partition of unity* construction a countable representation $\sum_{i=1}^{\infty} \psi_i(x)$. These partitions of unity allow us to split functions over charts in an atlas minimally for global to local constructions and via patching together local to global. More on this see [17].

From the point-set perspective, we want to distinguish how well certain types of subsets are "separated" with respect to the topology. Some of the situations we will consider can be summarized by the following picture which is abstracted from (pg.73 [10]):



Separation Axioms:

• Def $(\boldsymbol{X}, \mathscr{T})$ is a $\boldsymbol{T_1}$ **space** if points can be seperated v1:

$$\forall x, y \in X,\ \exists U \in \mathscr{T},\ \text{such that } x \notin U \text{ and } y \in U.$$

• Def $(\boldsymbol{X}, \mathscr{T})$ is a $\boldsymbol{T_2}$ **space** (a.k.a. **Hausdorff**) if points can be seperated v2:

$$\forall x, y \in X,\ \exists U, V \in \mathscr{T},\ \text{such that } x \in U,\ y \in V,\ \text{and } U \cap V = \emptyset.$$

• Def $(\boldsymbol{X}, \mathscr{T})$ is a $\boldsymbol{T_3}$ **space** if closed sets can be separated from their complements and if points can be separated.

Restating, $\boldsymbol{X}$ is $\boldsymbol{T_3}$ if it is $\boldsymbol{T_1}$ and:

$$\forall E_{closed} \subseteq X \text{ and } \forall x \in X \backslash E,\ \exists U_x, V_E \in \mathscr{T},\ \text{such that } U \cap V = \emptyset.$$

• Def $(\boldsymbol{X}, \mathscr{T})$ is a $\boldsymbol{T_4}$ **space** if closed disjoints sets are separable and $\boldsymbol{T_1}$. Again if:

$$\forall E_{closed}, F_{closed} \subseteq X,\ \text{with } E \cap F = \emptyset,\ \exists U_E, V_F \in \mathscr{T},\ \text{such that } U \cap V = \emptyset.$$

• Prop: (pg.73 [10]) $\boldsymbol{T_4} \implies \boldsymbol{T_3} \implies \boldsymbol{T_2} \implies \boldsymbol{T_1}$.

So, choosing a basis or generating set $\boldsymbol{\mathcal{B}}$ (resp. $\boldsymbol{S}$) such that its topological closure also satisfies one of the $\boldsymbol{T_i}$ axioms as well as being countable as a set is desirable.

Indeed, the property $\boldsymbol{T_2}$ is used in manifold theory, it implies uniqueness of limits of sequences $\boldsymbol{x} \in \boldsymbol{X}$ [Exercise: Prove this!] and $\boldsymbol{T_4}$ is used in Urysohn's Lemma and the Tietze Extension Theorem. These are important results [Exercise: See pgs.73-76 [10]]. Note the terms **Regular** and **Normal** are also used to respectively classify spaces with the properties seen in $\boldsymbol{T_3}$ and $\boldsymbol{T_4}$.

Quotient Topologies, Product and Coproduct Topologies

[Exercise: Use equivalence/congruence relations to partition spaces and give the corresponding topology. Define the topology corresponding to Cartesian product of sets and unions of sets. May need generation brackets $\langle \cdot \rangle_{\{\bigcup_{i\in I}, \bigcap_{i=1}^{N}\}}$ as in Comp Problem T P1 unions of topologies don't yield.]

Application of Topological Properties to Functions

I leave you with a list to be appended for tying together function domain and codomain properties such as open, closed, bounded, compact, complete, second-countable, separable (in the $\boldsymbol{T_i}$ sense), etc. A step in that direction comes from Lee's *Smooth Manifolds* Appendix A (p.596+ [14]).

• Prop (Closed Map Lemma): (p.610 [14]).

Suppose $\boldsymbol{X}$ is a compact space, $\boldsymbol{Y}$ is a Hausdorff space, and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a continuous map. Then:

a.) $\boldsymbol{f}$ maps closed sets to closed sets.

b.) If $\boldsymbol{f}$ is injective, it is a *topological embedding*.

c.) If $\boldsymbol{f}$ if bijective, it is a *homeomorphism*.

[Homeomorphisms are just bi-continuous bijective maps.]

• Def/Prop: P.s. **Disconnected spaces** can be written as the union of two or more disjoint sets. **connected spaces** are not disconnected. The only subsets of a *connected* topological space that are both open and closed are $\boldsymbol{\emptyset}, \boldsymbol{X}$ (pg.86 [10]). This is revisited in Comp Problem T P5.

THE END

[KTS. 12/05/21. 12:38am.]

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8. Results

<u>Sub-Table of Contents:</u>

The entries below, due to length of proof, would have interrupted the main content flow of the paper. So I pointed to them in cycles (in each entry there is a back arrow). The ordering is chronological, not matching the Table of Contents ordering. Lastly, the topics are random in a sense and don't necessarily have relevance to each other.

\ >> \ Jump to Comprehensive Exam Problems

This entry is from the discussion in Section 6.2.1

ENTRY 1:

Claim: Lebesgue measurability (i.e. in $\mathbb{R}^n$) is preserved under the trio of set operations $\{\bigcup_{i=1}^{\infty}, \bigcap_{i=1}^{\infty}, (\cdot \backslash \cdot\cdot)\}$.

- -

Proof:

- Case "$\bigcup_{i=1}^{\infty}$":

Proceed by induction on the upper limit of the union operator.

Base Case $(n = 2)$***:***

Suppose E_1 and E_2 are measurable. By definition (this.(p.42)), we have that E_j is measurable if:

$$\forall \epsilon > 0, \exists O_{\epsilon, open} \supseteq E_j \text{ such that } m_*(O_\epsilon \backslash E_j) \leq \epsilon.$$

So to define the corresponding family of open sets for each $\epsilon > 0$ for $E_1 \cup E_2$, it is natural to ask: What if we union the families for each E_1 and E_2? The answer lies in the *Set* property:

$$\forall X, Y, Z, W, \qquad (X \cup Y) \backslash (Z \cup W) \subseteq (X \backslash Z) \cup (Y \backslash W).$$

Which we prove for fun below:

- -

∘ **Lemma**: Consider the claim listed in black above. Let sets X, Y, Z, W be given and let $s \in (X \cup Y) \backslash (Z \cup W)$ be arbitrary. Logically we have:

$$[(s \in X) \vee (s \in Y)] \wedge \neg[(s \in Z) \vee (s \in W)].$$

By the Distribution Law for *conjunction* over *disjunction* operators $(P \vee Q) \wedge R \vDash (P \wedge R) \vee (Q \wedge R)$ and DeMorgan's Law $\neg(A \vee B) \vDash \neg A \wedge \neg B$, we get the above implies:

$$[(s \in X) \wedge ((s \notin Z) \wedge (s \notin W))] \vee [(s \in Y) \wedge ((s \notin Z) \wedge (s \notin W))].$$

By the Associative and Commutative Laws for conjunction, we can drop the inner parenthesis, rewrite to get X, Z and Y, W paired, and then add parentheses:

$$[((s \in X) \wedge (s \notin Z)) \wedge (s \notin W)] \vee [((s \in Y) \wedge (s \notin W)) \wedge (s \notin Z)].$$

Applying the Reduction Law $(P \wedge Q) \vDash P)$ and pulling out the negations implies:

$$[((s \in X) \wedge \neg(s \in Z))] \vee [((s \in Y) \wedge \neg(s \in W))].$$

But this says: $s \in (X \backslash Z) \cup (Y \backslash W)$.
By arbitrariness of s, we get the set containment: $(X \cup Y) \backslash (Z \cup W) \subseteq (X \backslash Z) \cup (Y \backslash W)$. □

- -

Proof (Continued):
Suppose E_1 and E_2 are measurable sets and that $\epsilon > 0$ is given.
Using the families of open sets provided by measurability of E_1 and E_2 (namely: $\{O_{1,\epsilon}\}_\epsilon$ and $\{O_{2,\epsilon}\}_\epsilon$), we have:

$$(O_{1,\epsilon} \cup O_{2,\epsilon}) \backslash (E_1 \cup E_2) \subseteq (O_{1,\epsilon} \backslash E_1) \cup (O_{2,\epsilon} \backslash E_2)$$

by the above Lemma. Then, Monotonicity of the (exterior) measure gives:

$$m_* \Big((O_{1,\epsilon} \cup O_{2,\epsilon}) \backslash (E_1 \cup E_2) \Big) \leq m_*(O_{1,\epsilon} \backslash E_1) + m_*(O_{2,\epsilon} \backslash E_2) \leq 2\epsilon.$$

So to complete the proof for the base case, we restate the result we obtained. For any $\epsilon > 0$, there exists an set $O_\epsilon := (O_{1,\frac{\epsilon}{2}} \cup O_{2,\frac{\epsilon}{2}}) \supseteq (E_1 \cup E_2)$, which is open since $\mathbb{R}^n$ is a *topological space*. Furthermore, we have: $m_*(O_\epsilon \backslash (E_1 \cup E_2)) \leq \epsilon$. Therefore $E_1 \cup E_2$ is (Lebesgue) measurable. □

Arbitrary Case ($n = k$) and Induction Step ($n = k + 1$):
Suppose for arbitrary $k \in \mathbb{N}$ the result holds. Then let $E := \bigcup_{i=1}^{k} E_i$, where all constituent E_i are measurable and let E_{k+1} be another measurable set. Then the new union:

$$\bigcup_{i=1}^{k+1} E_i := \left(\bigcup_{i=1}^{k} E_i \right) \cup E_{k+1}$$

is measurable by the induction hypothesis followed by the proof of the base case.

We conclude by our induction argument that countable unions, $E := \bigcup_{i=1}^{\infty} E_i$, of measurable sets are measurable. ■

- Case "$\bigcap_{i=1}^{\infty}$":

This proof rests on a similar Set theoretic Lemma:

$$\forall X, Y, Z, W, \qquad (X \cap Y) \backslash (Z \cap W) \subseteq (X \backslash Z) \cup (Y \backslash W).$$

Once this is proven, the rest is identical to the above case and we instead use $O_\epsilon := O_{1,\frac{\epsilon}{2}} \cap O_{2,\frac{\epsilon}{2}}$, quoting Monotonicity of the exterior measure to prove Lebesgue measurability of $\bigcap_{i=1}^{\infty} E_i$ by induction. A nuance here, that we overlook in the union case where it is more obvious, is that:

$$A \subseteq C \text{ and } B \subseteq D \implies A \cap B \subseteq C \cap D.$$

But this is easily provable (switch to F.O.L. statement and back). So $E_1 \cap E_2 \subseteq O_{1,\epsilon} \cap O_{2,\epsilon}$. ■

(Continues)

Proof (Continued (2)):

Attempting a direct proof for the last case leads to a rabbit hole, so we go with a rephrasing of the indirect route provided by (pg.18 of [19]) below.

- Case "$(\cdot \backslash \cdot\cdot)$":

We require 2 additional lemmas:

(1) Closed sets are measurable
(2) Sets of exterior measure zero are measurable.

[Exercise: Prove these!]

- -

The idea is, we split the complement into a union of two sets we can show are separately measurable. Then the result follows by the previous case for union above.

Suppose $\boldsymbol{E}$ is measurable. Then we have existence of a family of open sets $\{\boldsymbol{O}_{\epsilon}\}_{\epsilon>0}$ which all contain $\boldsymbol{E}$ and are such that $\boldsymbol{m}_{*}(\boldsymbol{O}_{\epsilon} \backslash \boldsymbol{E}) \leq \boldsymbol{\epsilon}$. We may construct from this family, a sub-family, $\{\boldsymbol{O}_{\frac{1}{i}}\}_{i \in \mathbb{N}}$, with countable index (i.e. a sequence). Hence we have $\boldsymbol{m}_{*}(\boldsymbol{O}_{\frac{1}{i}} \backslash \boldsymbol{E}) \leq \frac{1}{i}$ for every $\boldsymbol{i} \in \mathbb{N}$.

The further out the index goes, the more the open set constricts around $\boldsymbol{E}$. Likewise, the further out the index goes, the more the complement of the open set builds up to the complement of $\boldsymbol{E}$.

Define the limit of the sequence of the complements:

$$\boldsymbol{S} := \bigcup_{i=1}^{\infty} \boldsymbol{O}_{\frac{1}{i}}^{c}.$$

The complement of an open set is closed by definition and by (1), closed sets are measurable. By the countable union case we proved previously then we get $\boldsymbol{S}$ is measurable.

Since $\boldsymbol{x} \in (\boldsymbol{E}^{c} \backslash \boldsymbol{S})$ implies by DeMorgan's Law and Commutativity:

$$[(\boldsymbol{x} \notin \boldsymbol{O}_{1}^{c}) \wedge (\boldsymbol{x} \notin \boldsymbol{O}_{\frac{1}{2}}^{c}) \wedge \ldots] \wedge (\boldsymbol{x} \notin \boldsymbol{E}),$$

which then by Reduction and Negation yields: $(\boldsymbol{x} \in \boldsymbol{O}_{\frac{1}{i}}) \wedge \neg(\boldsymbol{x} \in \boldsymbol{E})$. This says $\boldsymbol{x} \in (\boldsymbol{O}_{\frac{1}{i}} \backslash \boldsymbol{E})$ so $(\boldsymbol{E}^{c} \backslash \boldsymbol{S}) \subseteq (\boldsymbol{O}_{\frac{1}{i}} \backslash \boldsymbol{E})$ by arbitrariness of $\boldsymbol{x}$. Monotonicity and arbitrariness of $\boldsymbol{i} \in \mathbb{N}$ yields $\boldsymbol{m}_{*}(\boldsymbol{E}^{c} \backslash \boldsymbol{S}) \leq \boldsymbol{m}_{*}(\boldsymbol{O}_{\frac{1}{i}} \backslash \boldsymbol{E}) \leq$ and by lemma (2), we have $\boldsymbol{E}^{c} \backslash \boldsymbol{S}$ is measurable. Since $\boldsymbol{S} \subseteq \boldsymbol{E}^{c}$, $\boldsymbol{E}^{c} = (\boldsymbol{E}^{c} \backslash \boldsymbol{S}) \cup \boldsymbol{S}$, we have $\boldsymbol{E}^{c}$ is measurable by previous base case for union.

Lastly, we note that $(\cdot \backslash \cdot\cdot) := (\cdot) \cap (\cdot\cdot)^{c}$, so we combine case proofs above and we're done. ■

This entry is from the discussion in Section 6.2.1

ENTRY 2: (Based off of Thm 3.2, pg.19 [19])

Claim: If $\boldsymbol{E}$ is a countable collection of disjoint measurable sets (that is $\boldsymbol{E} = \coprod_{i=1}^{\infty} \boldsymbol{E_{i,meas.}}$), then

$$\boldsymbol{m(E) = \sum_{i=1}^{\infty} m(E_i)}.$$

In other words, the Lebesgue measure satisfies **Countable Additivity**.

There are two directions for the proof $(\leq)$ *and* $(\geq)$.

The first follows from Sub-Additivity of $\boldsymbol{m_*}$ *(which we'll prove first and doesn't require disjointness). We extend this to the Lebesgue measure ,* $\boldsymbol{m}$*, by inheritance.*

The second part, follows from the result holding for disjoint, compact sets. We reduce to this case by assuming boundedness and using measurability of the complement, together with the Heine-Borel Theorem in $\mathbb{R}^{\boldsymbol{n}}$*. The following is based off of (pgs.13-19 [19]):*

Proof:

Case $(\leq)$ **:** Accordingly, let us prove $\boldsymbol{m_*}$ satisfies countable sub-additivity.

The inequality comes from the infimum definition "greatest (lower bound)". In particular, for any countable covering by closed cubes, $\boldsymbol{\mathcal{C} = \bigcup_{k=1}^{\infty} C_k}$, of a set $\boldsymbol{E}$, we have $\boldsymbol{m_*(E) \leq \sum_{k=1}^{\infty} |C_k|}$.

Now suppose $\boldsymbol{E := \bigcup_{i=1}^{\infty} E_i}$, we may find countable coverings for each of the sets $\boldsymbol{E_i}$ and union them together to get a cover for $\boldsymbol{E}$. This family of instances of covers (over choices for each $\boldsymbol{i}$),

$$\boldsymbol{\mathcal{C} := \bigcup_{i=1}^{\infty} \mathcal{C}_i := \bigcup_{i=1}^{\infty} \Big(\bigcup_{j=1}^{\infty} C_{i,j} \Big)}.$$

By definition of infimum, $\boldsymbol{m_*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i,j=1}^{\infty} |C_{i,j}| = (\bigstar)}$.

Now, this true for arbitrary choices of covers $\boldsymbol{\mathcal{C}_i \supseteq E_i}$ (arbitrary families) allows us to select a family of covers whose constituents have volume discrepency that adds together in a nice way.

Define $\boldsymbol{\mathcal{C}_i^{\epsilon}}$, indexed by $\boldsymbol{i} \in \mathbb{N}$ and $\boldsymbol{\epsilon > 0}$, such that $\boldsymbol{\sum_{j=1}^{\infty} |C_{i,j}^{\epsilon}| = m_*(E_i) + \frac{\epsilon}{2^i}}$.

Picking up where we left off with this family of covers applied:

$$\boldsymbol{(\bigstar) = \sum_{i=1}^{\infty} (m_*(E_i) + \frac{\epsilon}{2^i}) = \sum_{i=1}^{\infty} m_*(E_i) + \Big(\sum_{i=1}^{\infty} \frac{1}{2^i} \Big) * \epsilon = \sum_{i=1}^{\infty} m_*(E_i) + \epsilon}$$

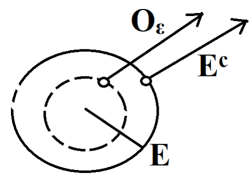
Restating, $\boldsymbol{\forall \epsilon > 0}$, $\boldsymbol{m_* \Big(\bigcup_{i=1}^{\infty} E_i \Big) \leq \sum_{i=1}^{\infty} m_*(E_i) + \epsilon}$. This implies the result. Since the Lebesgue measure is just a restriction of the exterior measure, it inherits this property. □

|<<| |<|

Proof (Continued):

***Case* $(\geq)$:**

For the other direction let use intuition from the following:



Previous work has shown that given a measurable set, E, we have E^c is also measurable. Hence for a given $\epsilon > 0$, we have existence of a family of open sets whose set difference has $m_*(O_\epsilon \backslash E^c) \leq \epsilon$.

Note that $O_\epsilon \backslash E^c = O_\epsilon \cap E = E \cap O_\epsilon = E \backslash O_\epsilon^c$.

This gives us a family of closed sets $\{O_\epsilon^c\}_\epsilon$ such that $m_*(E \backslash O_\epsilon^c) \leq \epsilon$ or in other words,

$$(m_*(E) - \epsilon) \leq m_*(O_\epsilon^c).$$

If it happens to be the case that E is bounded, by the **Heine-Borel Theorem in** $\mathbb{R}^n$, this implies the O_ϵ^c are all compact.

- -

Proceed by induction on the upper limit of the union operator.

Base case:
Suppose $E := E_1 \amalg E_2$ and both E_j are measurable and bounded. By the above, we have existence of compact subsets, $O_{1,\epsilon}^c$ and $O_{2,\epsilon}^c$, for arbitrary $\epsilon > 0$ (which must be disjoint) and such that $m_*(E_j) - \epsilon \leq m_*(O_{j,\epsilon}^c)$. Assuming super-additivity holds in the compact base case gives:

$$m_*(O_{1,\epsilon}^c \amalg O_{2,\epsilon}^c) \geq m_*(O_{1,\epsilon}^c) + m_*(O_{2,\epsilon}^c)$$

and hence by containment of $O_{1,\epsilon}^c \amalg O_{2,\epsilon}^c \subseteq E_1 \amalg E_2$ apply Monotonicity and the measure inequalities we obtained to get:

$$m_*(E_1 \amalg E_2) \geq m_*(E_1) + m_*(E_2) - 2\epsilon.$$

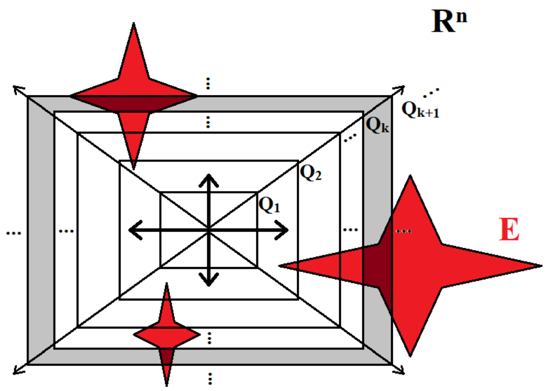
By arbitrariness of ϵ the result follows.

Induction Step:
Assume we have a disjoint union of length $n = k$ where each E_j is measurable and bounded and suppose the result holds for this union. We have by previous work that the union of measurable sets is measurable and through contrivance, one can show the finite union of bounded sets is bounded, so we effectively reduce to the **base case** when we union another appropriate E_{k+1}. By induction then, we have super-additivity in the case where the union components are further assumed *bounded*. Since $E_1, E_2,$ and $E_1 \amalg E_2$ are measurable, we can replace m_* with m.

- -

Proof (Continued (2)):

In the general case, define an increasing sequence of closed cubes in $\mathbb{R}^n$ as in the figure below.



We have for any set, $\boldsymbol{E = \bigcup_{k=1}^{\infty} (E \cap (Q_{k+1} \backslash Q_k))}$.

Clearly, each $\boldsymbol{(E \cap (Q_{k+1} \backslash Q_k))}$ is bounded.

So if $\boldsymbol{E := \bigcup_{i=1}^{\infty} E_i}$, we can rewrite:

$$\boldsymbol{E = \bigcup_{i,k=1}^{\infty} (E_i \cap (Q_{k+1} \backslash Q_k))}$$

Therefore, we can reduce to the case where all union components are measurable, disjoint, and bounded.

- -

It remains to reconcile our assumption of *super-additivity in the compact case.*

The idea is that any covering, $\boldsymbol{\{C_i\}_{i \in \mathbb{N}}}$, by closed cubes of the disjoint union, $\boldsymbol{A \amalg B}$ can be refined into a covering, $\boldsymbol{\{C'_j\}_{j \in \mathbb{N}}}$, of closed cubes which are all mutually disjoint (up to their boundaries) and exclusively contain subsets of $\boldsymbol{A}$ or $\boldsymbol{B}$ but not both. Then, using the compactness hypothesis, any such refinement yields a finite sub-covering $\boldsymbol{\{C''_k\}_{k \in \{1,\ldots,L\}}}$.

By finiteness, the total volume can be written:

$$\sum_{k=1}^{L} |C''_k| = \sum_{k_A=1}^{L_A} |C''_{k_A}| + \sum_{k_B=1}^{L_B} |C''_{k_B}|,$$

where $L = L_A + L_B$, each k_A and k_B are equal to some $k \in \{1,..,L\}$, and as suggested by the subscripts, $\boldsymbol{C''_{k_A}}$ contains some subset of $\boldsymbol{A}$ etc. The infimum value for this quantity is reached when both summands are at infimum seperately. So we proved equality and the result follows. ■

This entry is from the discussions in Section 2 and Section 4.2.

ENTRY 3:

Claim: $\mathbb{R}^n$ is complete with respect to the product of the *standard metric* on $\mathbb{R}$:

$$d_{std}(x, y) = |x - y|.$$

Proof: Summary:
(I) First, we use the Dedekind formulation of $\mathbb{R}$ to assert the **Completeness Property**, which says every non-empty subset of $\mathbb{R}$ bounded from above has a supremum that exists in $\mathbb{R}$,
[Def: p.22 [15]].
(II) Then, we show–using a Lemma–that Cauchy sequences in $(\mathbb{R}, d_{std})$ converge.
(III) Lastly, we show the property of being a *complete metric space* passes through Cartesian product in an inductive proof.

- -

I.) Accordingly, recall that a Dedekind cut, $\alpha \subseteq \mathbb{Q}$, satisfies:

(i) $\emptyset \neq \alpha \neq \mathbb{Q}$ [Proper Subset of $\mathbb{Q}$]

(ii) $\forall a \in \alpha,\ \forall b \in \mathbb{Q},\ (b < a) \implies (b \in \alpha)$ [No Lower Bound]
[a.k.a. $inf(\alpha) = -\infty$.]

(iii) $\forall a \in \alpha,\ \exists b \in \alpha,\ a < b < sup(\alpha)$ [No Largest Rational]

- -

(Beware: In what follows, we use two different order structures $\leq$ on $\mathbb{Q}$ and $\subseteq$ on $\mathbb{R}$ as the set of Dedekind cuts, but use the same symbol: $\leq$).

Let $\emptyset \neq A \subseteq \mathbb{R}$. This makes A a set of subsets of $\mathbb{Q}$ ordered by set inclusion. Naturally then, the supremum will be the upper limit of the inclusions, which we obtain via union:

$$\beta := \bigcup_{\alpha \in A} \alpha \subseteq \mathbb{Q}.$$

i.) Since A is nonempty by assumption $\exists \alpha \in A$ satisfying (i). This shows $\beta \neq \emptyset$. The further assumption that A is bounded above gives us $\beta < \infty$. In other words $\beta \neq \mathbb{Q}$.

ii.) Now, let $a \in \beta$ and $b \in \mathbb{Q}$ be such that $b < a$. We have by the union, that there exists $\alpha \in A$ such that $a \in \alpha$. Since α satisfies (ii), we know $b \in \alpha$ and hence $b \in \beta$. Arbitrariness of a and b gives the result.

iii.) Let $a \in \beta$. As before, by the union, there exists $\alpha \in A$ such that $a \in \alpha$. Since α satisfies (iii), $\exists b \in \alpha$ (hence $\in \beta$) such that $a < b < sup(\alpha) \leq sup(\beta)$. And we have established β as a Dedekind cut.

By arbitrariness of A, we conclude $\mathbb{R}$ satisfies the Completeness Property. □

Note: $\beta = sup(A)$ w.r.t. the order on $\mathbb{R}$ so $sup(\beta) = sup(sup(A))$ has two different uses of "sup".

Proof (Continued):

The proof for (II) is inspired by (p.58-p.62 [15]).

∘ **Lemma:** In a poset $(P, \leq)$, the statement below holds:

$$\forall p, q, r, \ [(p \leq q \leq r) \wedge (p = r)] \rightarrow (p = q = r).$$

Proof: $(p = r) \leftrightarrow [(p \leq r) \wedge (r \leq p)]$. So, the conjuction in hypotheses yields $(r \leq p \leq q)$, which further implies $(r \leq q)$. Together with the hypothesis $(q \leq r)$, this gives the result. □

II.) Let $\{x_i\}_{i \in \mathbb{N}}$ be a Cauchy sequence in $(\mathbb{R}, d_{std})$. Then,

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \ \forall m, n > N, \ \text{we have } |x_m - x_n| < \epsilon.$$

Define for a fixed ϵ_0, the $N(\epsilon_0)$-tail:

$$T_{N(\epsilon_0)} := \left\{ x_i \mid i > N(\epsilon_0) \right\} \qquad \subseteq \{x_i\}_{i \in \mathbb{N}}.$$

The Cauchy condition can be restated as:

$$\forall x, y \in T_{N(\epsilon_0)}, \qquad |x - y| < \epsilon_0$$

If we then open the absolute value bars, we get two statements:

$$(-\epsilon_0 < x - y) \text{ and } (x - y < \epsilon_0)$$

$$\implies (-\epsilon_0 + y < x) \text{ and } (x < \epsilon_0 + y)$$

Fixing *any* $y_0 \in T_{N(\epsilon_0)}$ then provides us–by arbitrariness of x in the above statement–with upper and lower bounds on $T_{N(\epsilon_0)}$. Namely:

$$lowerBound := -\epsilon_0 + y_0 \qquad \text{and} \qquad upperBound := \epsilon_0 + y_0.$$

By definition of infimum and supremum (respectively *greatest lower bound* and *least upper bound*), we can infer:

$$-\epsilon_0 + y_0 \leq \inf(T_{N(\epsilon_0)}) \qquad \text{and} \qquad \sup(T_{N(\epsilon_0)}) \leq \epsilon_0 + y_0$$

Negating the left statement, we get $-\inf(T_{N(\epsilon_0)}) \leq \epsilon_0 - y_0$. "Adding the inequalities", we arrive at:

$$\sup(T_{N(\epsilon_0)}) - \inf(T_{N(\epsilon_0)}) \leq 2\epsilon_0.$$

(Continues)

Proof (Continued (2)):

Arbitrariness of ϵ_0, says the above statement is true for all $\epsilon > 0$. Together with the fact that $0 \leq sup(P) - inf(P)$ for any poset P, in the limit as $\epsilon \to 0$, we get (by the above Lemma):

$$\lim_{N(\epsilon)\to\infty} \left\{ \inf\left(T_{N(\epsilon)}\right) \right\}_{N(\epsilon)\in\mathbb{N}} = \lim_{N(\epsilon)\to\infty} \left\{ \sup\left(T_{N(\epsilon)}\right) \right\}_{N(\epsilon)\in\mathbb{N}}.$$

Which is equivalent by definition to the statement:

$$\liminf\{x_i\}_{i\in\mathbb{N}} = \limsup\{x_i\}_{i\in\mathbb{N}}.$$

Since in general:

$$\liminf\{x_i\}_{i\in\mathbb{N}} \leq \lim_{i\to\infty}\{x_i\}_{i\in\mathbb{N}} \leq \limsup\{x_i\}_{i\in\mathbb{N}}$$

and we have for Cauchy sequences in (R, d_{std}), the endpoints are equal, we have (again by the Lemma):

$$\liminf\{x_i\}_{i\in\mathbb{N}} = \lim_{i\to\infty}\{x_i\}_{i\in\mathbb{N}} = \limsup\{x_i\}_{i\in\mathbb{N}}.$$

This *suggests* the sequence $\{x_i\}_{i\in\mathbb{N}}$ converges to (without loss of generality) $L := \limsup\{x_i\}_{i\in\mathbb{N}}$. However, it remains to be justified that $L \in \mathbb{R}$. We do this next.

Since Cauchy gave us an upper bound on all the $T_{N(\epsilon_0)}$, The Completeness Property of the Reals gives us all the $\sup(T_{N(\epsilon_0)}) \in \mathbb{R}$. In a proof similar to what we did in (I), instead using the intersection to define β, we have lower boundedness of the $T_{N(\epsilon_0)}$'s also give us $\inf(T_{N(\epsilon_0)}) \in \mathbb{R}$. So the sequences:

$$\left\{ \inf\left(T_{N(\epsilon)}\right) \right\}_{N(\epsilon)\in\mathbb{N}} \quad \text{and} \quad \left\{ \sup\left(T_{N(\epsilon)}\right) \right\}_{N(\epsilon)\in\mathbb{N}} \quad \subseteq \mathbb{R}.$$

But we have more. In particular, choosing ϵ_0 such that $N(\epsilon_0) = 1$, we have bounds on the entire sequence $\{x_i\}_{i\in\mathbb{N}}$ given by:

$$\widetilde{lowerBound} = min\{x_1, -\epsilon_0 + x_2\} \quad \text{and} \quad \widetilde{upperBound} = max\{x_1, \epsilon_0 + x_2\}.$$

So we have the subsequences of inf's and sup's are bounded as well.

Since decreasing the amount of elements in a set drives infimums up and supremums down, the sequences of inf's and sup's are respectively *increasing / decreasing* (not necessarily strictly). Hence, the respective limits of the subsequences are given by:

$$\sup\left(\left\{ \inf\left(T_{N(\epsilon)}\right) \right\}_{N(\epsilon)\in\mathbb{N}} \right) \quad \text{and} \quad \inf\left(\left\{ \sup\left(T_{N(\epsilon)}\right) \right\}_{N(\epsilon)\in\mathbb{N}} \right)$$

which are both equal to L. By the Completeness Property, we conclude $L \in \mathbb{R}$. □

(Continues)

Proof (Continued (3)):

Since the other direction is proved in Section 4.2 (namely Convergent implies Cauchy), we conclude (R, d_{std}) is a complete metric space. ■

III.) In the final segment of the proof, we must show that the *direct product* of any finite number of complete metric spaces is complete.

Proceed by induction on the number of elements in the direct product.

Base Case ($n = 2$):

Suppose (A, d_A) and (B, d_B) are complete metric spaces. The product space is given by:

$$C := A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Moreover, the function:

$$d_C : C \times C \to \mathbb{R}^{\geq 0}$$
$$\big((a_1, b_1), (a_2, b_2)\big) \mapsto d_A(a_1, a_2) + d_B(b_1, b_2),$$

yields a metric on C. [Exercise: Prove this! Easy.]

Let $\{(a_i, b_i)\}_{i\in\mathbb{N}}$ be a Cauchy sequence in C. By definition:

$$\forall \epsilon,\ \exists N_\epsilon,\ \forall m, n > N_\epsilon,\ d_C\big((a_m, b_m), (a_n, b_n)\big) < \epsilon$$

Maintaining the quantifiers, this implies:

$$d_A(a_m, a_n) + d_B(b_m, b_n) < \epsilon$$

But since both d_A and d_B are positive definite, we can go left on the inequality and retrieve seperately:

$$d_A(a_m, a_n) < \epsilon \qquad \text{and} \qquad d_B(b_m, b_n) < \epsilon.$$

This makes $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ Cauchy sequences in (A, d_A) and (B, d_B) respectively. Since the component spaces are complete, we know these sequences converge to elements $a \in A$ and $b \in B$. Writing out the definition of convergence in both cases yields:

$$\forall \epsilon > 0,\ \exists N_\epsilon := max\{N_{\epsilon/2,A}, N_{\epsilon/2,B}\},\ \forall i > N_\epsilon,\ d_C\big((a_i, b_i), (a, b)\big) = d_A(a_i, a) + d_B(b_i, b) < \epsilon.$$

Hence $\{(a_i, b_i)\}_{i\in\mathbb{N}}$ converges to $(a, b) \in A \times B$ making C complete. □.

(Continues)

Proof (Continued (4)):

Induction Step (n=k+1):

Suppose the result holds for $\boldsymbol{n = k}$ and take complete metric spaces $(\boldsymbol{A_j, d_{A_j}})$ for $\boldsymbol{j \in \{1, ..., k+1\}}$. Using the induction hypothesis and the base case, we have the following metric space is complete:

$$\Big((\boldsymbol{A_1 \times ... \times A_k}) \times \boldsymbol{A_{k+1}}, \qquad \boldsymbol{d_{A_1 \times ... \times A_k} + d_{A_{k+1}}} \Big).$$

But this is up to identification, the space:

$$\Big(\boldsymbol{A_1 \times ... \times A_{k+1}}, \qquad \boldsymbol{d_{A_1 \times ... \times A_{k+1}}} \Big).$$

Take a Cauchy sequence in the second space, identify it with a Cauchy sequence in the first space. Back identify the limit we obtain by completeness. Then: $\boldsymbol{\forall \epsilon, \exists N_\epsilon, \ \forall i > N_\epsilon,}$

$$\boldsymbol{d_{A_1 \times ... \times A_{k+1}} \Big((a_{i,1}, ..., a_{i,k+1}), (a_1, ..., a_{k+1}) \Big) = d_{A_1 \times ... \times A_k} \Big((a_{i,1}, ..., a_{i,k}), (a_1, ..., a_k) \Big)}$$
$$\boldsymbol{+ d_{A_{k+1}}(a_{i,k+1}, a_{k+1}) < \epsilon}.$$

Thus the result is proven for $\boldsymbol{n = k+1}$.

The conclusion of induction then gives the $\boldsymbol{n}$-fold direct product of complete metric spaces is complete, for any $\boldsymbol{n} \in \mathbb{N}$. □

- -

We conclude (III) with the immediate Corollary of the result proven above:

$(\boldsymbol{R^n, d^n_{std.}})$ with the standard product metric is complete as a metric space for any $\boldsymbol{n} \in \mathbb{N}$. ■

Remarks: This proof should be analyzed and abstracted wherever possible to provide a route for proving completeness of arbitrary metric spaces (with order structures).

This entry is from the discussion on Subspace and Induced Topologies in Section 7.1.

ENTRY 4:

Claim: Injective Set Maps Preserve Intersection and Union. That is, for $\iota : X \hookrightarrow Y$, we have $\iota(A \cap B) = \iota(A) \cap \iota(B)$ and $\iota(A \cup B) = \iota(A) \cup \iota(B)$ for any $A, B \subseteq X$.

Proof: Proof goes by bi-directional containment and opening up the set descriptions logically. Let $A, B \subseteq X$.

($\subseteq$) :

WTS $\iota(A \cap B) \subseteq \iota(A) \cap \iota(B)$. Accordingly, let $y \in \iota(A \cap B)$, then $\exists x \in A \cap B$ such that $y = \iota(x)$. By definition, $x \in (A \cap B)$ implies $(x \in A)$ and $(x \in B)$. So $\big(\iota(x) \in \iota(A)\big)$ and $\big(\iota(x) \in \iota(B)\big)$. Hence again by definition of the intersect set, $y = \iota(x) \in \big(\iota(A) \cap \iota(B)\big)$. Arbitrariness of $y \in \iota(A \cap B)$ yields the desired containment.

($\supseteq$) :

Conversely, suppose $y \in \big(\iota(A) \cap \iota(B)\big)$. Then $y \in \iota(A)$ and $y \in \iota(B)$. These two imply $\exists a \in A$ and $\exists b \in B$ such that $\iota(a) = y = \iota(b)$. By hypothesis $\iota : X \hookrightarrow Y$ is injective, so $\iota(a) = \iota(b)$ implies $a = b$. Thus $a \in B$ as well. So $a \in A \cap B$, which makes $y = \iota(a) \in \iota(A \cap B)$. Arbitrariness of $y \in \big(\iota(A) \cap \iota(B)\big)$ gives $\iota(A \cap B) \supseteq \iota(A) \cap \iota(B)$.

We conclude $\iota(A \cap B) = \iota(A) \cap \iota(B)$. □

- -

For the union atom, we don't need the injective hypothesis, it is true for set maps in general as we will show:

($\subseteq$) :

Let $y \in \iota(A \cup B)$. Then $\exists x \in (A \cup B)$ with $y = \iota(x)$. By definition of union, $x \in A$ or $x \in B$. Without loss of generality due to Commutativity of union, suppose the first case is true. Then $y = \iota(x) \in \iota(A) \subseteq \big(\iota(A) \cup \iota(B)\big)$. Arbitrariness of y yields the result.

($\supseteq$) :

Conversely, let $y \in \big(\iota(A) \cup \iota(B)\big)$. Then by definition $y \in \iota(A)$ or $y \in \iota(B)$. WLOG, take $y \in \iota(A)$ to be true. Then $\exists x \in A$ such that $y = \iota(x)$. But $x \in A \subseteq (A \cup B)$.
So $y = \iota(x) \in \iota(A \cup B)$. Arbitrariness of y finishes. We conclude $\iota(A \cup B) = \iota(A) \cup \iota(B)$. ■

9. Comp Problems and Select Solutions

Sub-Table of Contents: (Choose six of each)

Real Analysis Comprehensive Exam Problems (See CSULB Spring 2021):

Topology Comprehensive Exam Problems (See CSULB Spring 2021):

RA P1: Sigma Algebras and Borel Sets

Statement:
a.) Let $\mathcal{A}$ be the σ-algebra generated by all sets in $\mathbb{R}$ of the form $[a, \infty)$. Let $\mathcal{B}$ be the σ-algebra of Borel sets in $\mathbb{R}$. Prove $\mathcal{A} = \mathcal{B}$.

b.) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Assume for all $a \in \mathbb{R}$, $f^{-1}\big([a, \infty)\big)$ is a Borel set. Prove that for any Borel set B, $f^{-1}(B)$ is a Borel set.

- -

Proof:

a.)
Preliminary Note 1, for (a):
We are proving equality of two σ-algebras that we happen to know the generators of. Recall the definition of σ-algebra: A set closed under $\mathscr{O} := \left\{ \bigcup_{i \in I}, \bigcap_{j \in J}, (\cdot)^c \right\}$.

If we can show $\forall G_{generators} \in \mathcal{A} : \ G \in \mathcal{B}$, then $\mathcal{A} := \langle G \rangle_{\mathscr{O}} \subseteq \mathcal{B}$ as $\mathcal{B}$ is a σ-algebra. Hence it would be the case that $\mathcal{A} \subseteq \mathcal{B}$.

The other direction is symmetric and requires showing $\forall H_{generators} \in \mathcal{B} : \ H \in \mathcal{A}$. Combining these two results gives $\mathcal{A} = \mathcal{B}$.

In our instance, we have:

$$\mathcal{A} := \langle \ \{G := [a, \infty) \mid a \in \mathbb{R}\} \ \rangle_{\mathscr{O}} \qquad \text{and} \qquad \mathcal{B} := \langle \ \{H := O_{open} \subseteq \mathbb{R}\} \ \rangle_{\mathscr{O}}.$$

The second is by definition of *Borel Sets.*

- -

Preliminary Note 2, for (a):
Since we also know $\mathbb{R}$ can be given metric space structure with at least $(\mathbb{R}, d_{std.})$, we may use the metric topology and corresponding definition of open in terms of union of open metric balls, $B_r(x)$, which looks like:

$$O_{open} = \bigcup_{i \in I} B_{r_i}(x_i) = \bigcup_{i \in I} (a_i, b_i) \qquad << -- \ \text{Intervals}$$

in the standard metric $d(x, y) = |y - x|$.

Proof of (a):
$(\subseteq)$ **:** Let $[a, \infty) \in \mathcal{A}$ be given. Since $[a, \infty) = \big((-\infty, a)\big)^c = \big(\bigcup_{i \in \mathbb{N}} (a - i, a)\big)^c$, we see that $[a, \infty)$ can be written as the compliment of an open set (Prelim. Note 2) in $(\mathbb{R}, d_{std.})$, so it is an element of the $\mathcal{B}$. Arbitrariness of $a \in \mathbb{R}$ yields the result by Note 1.

(Continues)

Proof (Continued):

$(\supseteq)$ **:** Let $O_{open} \subseteq (\mathbb{R}, d_{std.})$ be given. Then working backwards, we get $O = \bigcup_{i\in I}(a_i, b_i)$ by Note 2. This implies:

$$O = \bigcup_{i\in I}\Big((-\infty, a_i] \cup [b_i, \infty)\Big)^c$$
$$= \bigcup_{i\in I}\Big(((a_i, \infty))^c \cup [b_i, \infty)\Big)^c$$
$$= \bigcup_{i\in I}\Big((\bigcap_{j\in\mathbb{N}}[a_i + \frac{1}{j}, \infty))^c \cup [b_i, \infty)\Big)^c.$$

From which, we conclude $O \in \mathcal{A}$. By Note 1, arbitrariness of O gives us the reverse containment $\mathcal{A} \supseteq \mathcal{B}$. Therefore, $\mathcal{A} = \mathcal{B}$. ■

---

b.)

Recall the statement: Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Assume for all $a \in \mathbb{R}$, $f^{-1}\big([a, \infty)\big)$ is a Borel set. Prove that for any Borel set B, $f^{-1}(B)$ is a Borel set.

- -

Let $B \in \mathcal{B}$ as defined in (a). Then the result of (a) tells us we may write B as an algebraic combination of elements of $\mathcal{A}$ whom are generated by $\{[a, \infty) \mid a \in \mathbb{R}\}$.

If $f^{-1} : \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ is a **σ-algebra homomorphism** then the algebraic expression for B gives an algebraic expression for $f^{-1}(B)$ in terms of $f^{-1}([a, \infty))$ which we know are Borel by hypothesis (hence the expression for $f^{-1}(B)$ is in $\mathcal{B}$ (by closure of $\mathcal{B}$ under the operations). □

• We proceed to show f^{-1} is such a map (doesn't depend on $\mathbb{R}$ in general). By the atomic operations $\{\cup, \cap, (\cdot)^c\}$. We claim for arbitrary $S, T \subseteq \mathbb{R}$, that $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$, $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$, and $f^{-1}(X - S) = f^{-1}(X) - f^{-1}(S)$. But this is three similar proofs (left/right containment arguments in first-order logic). I'll do one, the rest are similar. [Exercise: Prove these.]

- -

Let $x \in f^{-1}(S \cup T)$ then $f(x) \in (S \cup T)$ which implies $(f(x) \in S) \vee (f(x) \in T)$. This implies $(x \in f^{-1}(S)) \vee (x \in f^{-1}(T))$ which finally gives $x \in \big(f^{-1}(S) \cup f^{-1}(T)\big)$. Arbitrariness of x gives "$(\subseteq)$".

Conversely, let $y \in \big(f^{-1}(S) \cup f^{-1}(T)\big)$ then $(y \in f^{-1}(S)) \vee (y \in f^{-1}(T))$, which implies $(f(y) \in S) \vee (f(y) \in T)$. Finally, we get $f(y) \in (S \cup T)$ and hence $y \in f^{-1}(S \cup T)$. Arbitrariness of y gives "$(\supseteq)$" and hence pre-image maps, f^{-1}, preserve unions. ■

RA P2: Laplace Transform of Measurable Function is Uniformly Continuous

Statement:
Assume $f \in L^1(0, \infty)$. Define

$$F(x) = \int_0^\infty e^{-xt} f(t) dt.$$

Prove that F is uniformly continuous for $0 \leq x < \infty$.

Proof:

RA P3: Convolution of Measurable Functions is Measurable

Statement:
Prove that for all Lebesgue integrable $\boldsymbol{f, g : \mathbb{R}^n \to \mathbb{R}}$ the convolution $\boldsymbol{f * g}$,

$$\boldsymbol{(f * g)(x) = \int_{-\infty}^{\infty} f(t - x)g(x)dx}$$

is Lebesgue integrable.

- -

Proof:

RA P4: Point-wise Convergence implies Convergence in Measure

Statement:
Convergence of the sequence f_n to f in measure on $(0,1)$ means that

$$\text{for all } \epsilon > 0, \quad \lim_{n\to\infty} \lambda\{x \in (0,1) \bigm| |f_n(x) - f(x)| \geq \epsilon\} = 0.$$

Show that convergence in measure of f_n to f on the interval $(0,1)$ is implied by pointwise convergence almost everywhere on $(0,1)$, that is, by $f_n(x) \to f(x)$ for almost every $x \in (0,1)$.

[λ *is the Lebesgue measure.]*

Proof:
Let $\epsilon > 0$ and consider for a given member of the family of functions, f_n, the associated set:

$$E_{\epsilon,n} := \{x \in (0,1) \bigm| f_n(x) \notin B_\epsilon(f(x))\}.$$

- -

There are two possibilities for any $x \in (0,1)$: Either $f_n(x) \to f(x)$ or $f_n(x) \not\to f(x)$ (either the sequence of images of x converges to $f(x)$ or it doesn't).

- -

In the first case, we know for our given ϵ there exists an N for which all $f_n(x) \in B_\epsilon(f(x))$ when $n > N$. This implies the x we start with is not a member of $E_{\epsilon,n}$ for any $n > N$, we deduce that

$$x \notin \lim_{n\to\infty} (E_{\epsilon,n}).$$

Arbitrariness of x in this case says $\lim_{n\to\infty} E_{\epsilon,n}$ may consist of only points, x, where $f_n(x) \not\to f(x)$.

- -

Applying the hypothesis then, if $f_n(x) \to f(x)$ a.e. on $(0,1)$, then the set of points, E, where point-wise convergence fails has measure zero.

But we have $\left(\lim_{n\to\infty} E_{\epsilon,n}\right) \subseteq E$, so Monotonicity implies:

$$\lim_{n\to\infty} \lambda(E_{\epsilon,n}) = \lambda\Big(\lim_{n\to\infty} E_{\epsilon,n}\Big) \leq \lambda(E) = 0$$

And the result follows by arbitrariness of $\epsilon > 0$. Restating: We showed p.w. convergence a.e. implies convergence in measure. Nothing was special about the domain being $(0,1)$. ■

RA P5: Trivia Regarding Integrability.

Statement:
Let f be a Lebesgue integrable function on $\mathbb{R}$. Prove that for any $L > 0$

$$\lim_{n\to\infty} \int_{n}^{n+L} f(x)dx = 0.$$

- -

Proof:
Assume for the moment that $f(x)$ is positive on all of $\mathbb{R}$. Then intuitively, if f is integrable, we have $I_f < \infty$ and so the limit of contributions to the integral from intervals $[n, n+L]$ must go to zero (otherwise the infinite sum of all of these positive contributions would be infinite, which is the desired contradiction since the integral is at least this value).

- -

Assume now the general case for f, then we may split $f = f^+ - |f^-|$, where $f^+(x) := max\{f(x), 0\}$ and $f^-(x) := min\{f(x), 0\}$.

Note if $f(x) > 0$ then $f^+(x) > 0$ and $f^-(x) = 0$. Likewise if $f(x) < 0$ then $f^+(x) = 0$ and $f^-(x) < 0$.

We proved the result holds for f^+ and $|f^-|$ since these are positive functions. Now finally:

$$\lim_{n\to\infty} \int_{[n,n+L]} f(x)dx = \lim_{n\to\infty} \int_{[n,n+L]} f^+(x) - |f^-(x)|dx$$

$$= \lim_{n\to\infty} \int_{[n,n+L]} f^+(x)dx - \lim_{n\to\infty} \int_{[n,n+L]} |f(x)|dx = 0$$

by linearity of the integral and limit operators. $L > 0$ arbitrary ensured the contribution argument worked. ■

RA P6: Positive Functions with Zero Integral are Zero A.E.

Statement:
a.) Suppose f is an integrable function on E and suppose that $f \geq 0$ on E. For $\alpha > 0$, define $E_\alpha = \{x \in E \mid f(x) > \alpha\}$. Prove that

$$\lambda(E_\alpha) \leq \frac{1}{\alpha} \int_E f.$$

[λ *is the Lebesgue measure.]*

b.) Suppose that $f \geq 0$ on E and $\int_E f = 0$. Prove that $f(x) = 0$ almost everywhere on E.

Proof:

a.) By definition, the value of $f(x)$ over E_α is greater than α, so we have:

$$\int_{E_\alpha} \alpha dx < \int_{E_\alpha} f(x) dx.$$

By Linearity of the integral, the left hand side equals

$$\alpha \int_{E_\alpha} dx =: \alpha \cdot \lambda(E_\alpha).$$

This implies:

$$\lambda(E_\alpha) < \frac{1}{\alpha} \int_{E_\alpha} f(x) dx \leq \frac{1}{\alpha} \int_E f(x) dx.$$

Where the last inequality follows from the fact that the integral must be larger, since we are adding on more positive contributions in the containing set E.

By Logical "Addition", $T \vDash T \vee \varphi$, we arrive at the Result by letting φ be the case for equality (we proved the statement involving less than is true). ■

(Continues)

Proof (Continued):

b.) Suppose $f \geq 0$ on E and that $\int_E f = 0$.

We have:

$$S := \{x \in E \mid f(x) > 0\} = f^{-1}\big((0,\infty)\big)$$
$$= f^{-1}\big(\bigcup_{n\in\mathbb{N}}(\frac{1}{n},\infty)\big)$$
$$= \bigcup_{n\in\mathbb{N}} f^{-1}\big((\frac{1}{n},\infty)\big) = \bigcup_{n\in\mathbb{N}} E_{1/n}.$$

- -

We next observe from part (a), that $f \geq 0$ on E implies in particular that:

$$\forall n \in \mathbb{N}(\subseteq \mathbb{R}), \qquad \lambda(E_{1/n}) \leq \frac{1}{1/n} \cdot \int_E f(x)dx.$$

By the second hypothesis of this problem, $\int_E f = 0$, so $\forall n \in \mathbb{N}$, $\lambda(E_{1/n}) \leq 0$.

- -

We now combine our results:

$$\lambda(S) = \lambda\Big(\bigcup_{n\in\mathbb{N}} E_{1/n}\Big) \leq \sum_{i\in\mathbb{N}} \lambda(E_{1/n}) \leq 0,$$

by Countable Sub-Additivity of λ. Since measures are also positive, we conclude $\lambda(S) = 0$. But points in S are the set of points where the property **"f(x)=0"** does not hold and it has zero measure. Therefore $f(x) = 0$ **a.e. on** E. ■

RA P7: Measure of the Rationals and Irrationals in $[0, 1]$

Statement:
a.) Show that the set of rational numbers in $[0, 1]$ is of measure 0.

b.) Show there is a closed subset $A \subset [0, 1]$ that contains no rational number such that $\lambda(A) > 0$.

[λ *is the Lebesgue measure.]*

Proof:

a.) WTS $\mathbb{Q} \cap [0, 1] \subseteq \mathbb{R}$ has (exterior) measure 0.

Recall the exterior measure of a subset $S \subseteq \mathbb{R}$ is given by:

$$m_* : \mathscr{P}(\mathbb{R}) \to [0, \infty]$$

$$S \mapsto \inf_{\mathcal{C}} \left\{ \sum_{i \in I} |C_i| : \text{ where } C_i \in \mathcal{C} \right\}$$

and the infimum is taken over coverings of S by closed cubes. $|C_i|$ being the volume of cube $C_i \in \mathcal{C}$.

○ Lemma 1: $\forall p \in \mathbb{R}$, $m_*(\{p\}) = 0$.

Proof: $\{p\}$ can be covered by a single cube (closed interval) of arbitrarily small side length (and hence of arbitrarily small volume).

Since all possible sums of volumes of cubes are positive numbers (≥ 0) and we just showed existence of elements with volumes going to zero the infimum of total volumes over the set of all coverings by closed cubes must be zero (since it lies in between these). □

By a well known result (zigzag array enumeration), the rationals are countably enumerable. That is:

$$\mathbb{Q} = \bigcup_{i \in \mathbb{N}} \{q_i\}.$$

We may use this, together with the properties of the exterior measure called Monotonicity and Countable Sub-Additivity to infer:

$$m_*(\mathbb{Q} \cap [0, 1]) \leq m_*(\mathbb{Q}) = m_* \left(\bigcup_{i \in \mathbb{N}} \{q_i\} \right) \leq \sum_{i \in \mathbb{N}} m_*(\{q_i\}) = 0$$

where the last equality is from Lemma 1. We conclude $m_*(\mathbb{Q} \cap [0, 1]) = 0$. ■

Proof (Continued):

b.) The Lebesgue measure $\boldsymbol{\lambda : \mathcal{M} \to [0, \infty]}$ is defined on a restricted collection of subsets of $\boldsymbol{E} \subseteq \mathbb{R}$ defined by the condition of Lebesgue measurability:

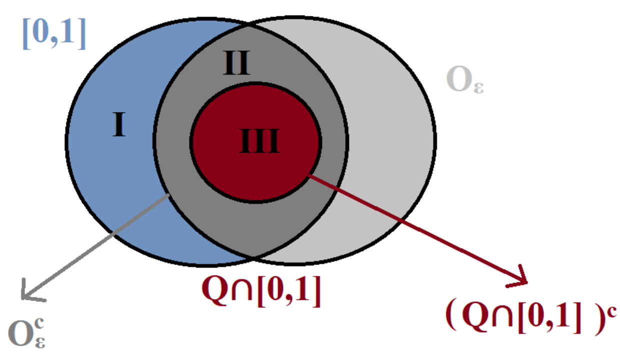
$$\text{“}\exists\{\boldsymbol{O_{\epsilon,open}}\}_{\epsilon>0} \text{ for which all } (\boldsymbol{O_\epsilon} \supseteq \boldsymbol{E}) \text{ and } \boldsymbol{m_*(O_\epsilon \backslash E) \leq \epsilon}\text{”}.$$

Sets that satisfy this condition have $\boldsymbol{\lambda(E) := m_*(E)}$. Moreover, assume $\boldsymbol{\mathcal{M}}$ forms a $\boldsymbol{\sigma}$-algebra and that sets of exterior measure zero are measurable, so that $\mathbb{Q} \cap [\mathbf{0}, \mathbf{1}]$ is measurable by (a).

- -

Using the definition of measurability of $\mathbb{Q} \cap [\mathbf{0}, \mathbf{1}]$, we have existence of a family of open sets, $\{\boldsymbol{O_\epsilon}\}_{\epsilon>0}$, each containing $\mathbb{Q} \cap [\mathbf{0}, \mathbf{1}]$ with $\boldsymbol{m_*}\big(\boldsymbol{O_\epsilon} \backslash (\mathbb{Q} \cap [\mathbf{0}, \mathbf{1}])\big) \leq \boldsymbol{\epsilon}$.

Considering the following crude picture suggests we split $[\mathbf{0}, \mathbf{1}]$ into a union with three components (for fixed epsilon).



$$[\mathbf{0}, \mathbf{1}] = \boldsymbol{I} \cup \boldsymbol{II} \cup \boldsymbol{III} := (\boldsymbol{O_\epsilon^c} \cap [\mathbf{0}, \mathbf{1}]) \cup \Big(\big(\boldsymbol{O_\epsilon} \backslash (\mathbb{Q} \cap [\mathbf{0}, \mathbf{1}])\big) \cap [\mathbf{0}, \mathbf{1}]\Big) \cup (\mathbb{Q} \cap [\mathbf{0}, \mathbf{1}])$$

Now, using Countable Sub-Additivity to split the union into a sum and Monotonicity applied to the second component, we get:

$$\boldsymbol{m_*}([\mathbf{0}, \mathbf{1}]) \leq \boldsymbol{m_*}(\boldsymbol{O_\epsilon^c} \cap [\mathbf{0}, \mathbf{1}]) + \boldsymbol{m_*}\big(\boldsymbol{O_\epsilon} \backslash (\mathbb{Q} \cap [\mathbf{0}, \mathbf{1}])\big) + \boldsymbol{m_*}(\mathbb{Q} \cap [\mathbf{0}, \mathbf{1}])$$

But we know $\boldsymbol{m_*}([\mathbf{0}, \mathbf{1}]) = \mathbf{1},\ \boldsymbol{m_*}(\mathbb{Q} \cap [\mathbf{0}, \mathbf{1}]) = \mathbf{0},$ and $\boldsymbol{m_*}\big(\boldsymbol{O_\epsilon} \backslash (\mathbb{Q} \cap [\mathbf{0}, \mathbf{1}])\big) \leq \boldsymbol{\epsilon}$. So:

$$\boldsymbol{m_*}(\boldsymbol{O_\epsilon^c} \cap [\mathbf{0}, \mathbf{1}]) \geq \mathbf{1} - \boldsymbol{\epsilon}.$$

And we have control of $\boldsymbol{\epsilon} > \mathbf{0}$, so let us choose $\boldsymbol{\epsilon_0} \in (\mathbf{0}, \mathbf{1})$ and define:

$$\boldsymbol{A} := \boldsymbol{O_{\epsilon_0}^c} \cap [\mathbf{0}, \mathbf{1}].$$

We have $\boldsymbol{O_{\epsilon_0}} \supseteq \mathbb{Q}$ implies $\boldsymbol{O_{\epsilon_0}^c} \subseteq \mathbb{Q}^c$ so that $\boldsymbol{A} \cap \mathbb{Q} = \boldsymbol{\emptyset}$, clearly $\boldsymbol{A} \subseteq [\mathbf{0}, \mathbf{1}]$, and $\boldsymbol{A}$ is closed since $\boldsymbol{O_{\epsilon_0}}$ is open, $[\mathbf{0}, \mathbf{1}]$ is closed and so the desired intersection is closed. Lastly, open sets are measurable, and $\boldsymbol{\mathcal{M}}$ is a $\boldsymbol{\sigma}$-algebra, so $\boldsymbol{A}$ is Lebesgue measurable with $\boldsymbol{\lambda(A) = m_*(A) \geq 1 - \epsilon_0 > 0}$ as we've shown. ■

RA P8: Unnamed

Statement:
Let $g(x) = \frac{1}{x^{1/3}}\chi_{(0,1)}(x)$, and let r_n be an enumeration of the rational numbers in $\mathbb{R}$.

Let $f(x) = \sum_{i=1}^{\infty} \frac{1}{n^2} g(x - r_n)$. Show that $f \in L^1(\mathbb{R})$ and that $f(x)$ is finite almost everywhere.

[$\chi_A(x)$ *is the characteristic function that equals* 1 *for* $x \in A$ *and* 0 *for* $x \notin A$.]

- -

Proof:

RA P9: Inequalities for Different p-Norms

Statement:
Suppose that $0 < p < q < r < \infty$, and suppose that $\int_{\mathbb{R}} |f|^p < \infty$ and $\int_{\mathbb{R}} |f|^r < \infty$. Prove that $\int_{\mathbb{R}} |f|^q < \infty$.

Proof: If $0 < q < r$ then by the Euclidean Algorithm, $\exists k \in \mathbb{N}$ and $s \in [0, q)$ such that:

$$r = kq + s$$

From this, we get for any positive real $A \in \mathbb{R}^+$,

$$A^r = (A^q)^k \cdot A^s$$

by ancient results from basic algebra. Now, RHS $\geq (A^q)^k$ with equality only when $s = 0$ and subsequently this is $\geq A^q$ with equality only when $k = 1$. We deduce:

$$A^r > A^q$$

since we know we can't have both $s = 0$ and $k = 1$ (which would say $r = q$).

Applying the above result to $A := |f(x)|$ for any $x \in \mathbb{R}$ says:

$$\forall x \in \mathbb{R} : \qquad |f(x)|^q < |f(x)|^r.$$

Applying the integrals then:

$$\int_{\mathbb{R}} |f(x)|^q dx < \int_{\mathbb{R}} |f(x)|^r dx.$$

Finally if we suppose that RHS is finite, we're done. This is independent of values associated to p. ■

T P1: Unions and Intersections of Topologies

Statement:
Let τ_α for each $\alpha \in A$ be a topology on $\mathbb{R}$.

a.) Prove or provide a counter example to the statement: $\bigcup_{\alpha \in A} \tau_\alpha$ is a topology on $\mathbb{R}$.

b.) Prove or provide a counter example to the statement: $\bigcap_{\alpha \in A} \tau_\alpha$ is a topology on $\mathbb{R}$.

- -

Proof:
a.) Define two topologies on $\mathbb{R}$ by:

$$\tau_1 := \langle \{ [a, \infty) \mid a \in \mathbb{R} \} \rangle \cup \{\emptyset, \mathbb{R}\}$$

and

$$\tau_2 := \langle \{ (-\infty, b] \mid b \in \mathbb{R} \} \rangle \cup \{\emptyset, \mathbb{R}\},$$

where the outer angle brackets denote generation by the operations $\bigcup\limits_{i \in I}$ and $\bigcap\limits_{i=1}^{N}$.

Consider the union: $\bigcup_{\alpha \in A} \tau_\alpha := \tau_1 \cup \tau_2$.

We may break the algebraic closure of $\tau_1 \cup \tau_2$ with the following counter example:

$$U := [a, \infty) \quad \text{and} \quad V := (-\infty, b] \quad \text{for } a < b.$$

$$\implies U \cap V = [a, b].$$

$[a, b]$ is neither an element of τ_1 nor of τ_2 (since it doesn't have an infinite head or tail on the defining condition). So $U \cap V \notin \tau_1 \cup \tau_2$, but $U, V \in \tau_1 \cup \tau_2$ by identification $\tau_i \hookrightarrow \tau_1 \cup \tau_2$.

We conclude: *unions of topologies are not (in general) topologies.*

[×]

(Next page)

Proof (Continued):

b.) WTS $\bigcap_{\alpha\in A}\tau_\alpha$ is a topology whenever all τ_α are.

First let's consider the base case. Let τ_1 and τ_2 be topologies on a set X. Then, suppose $\{U_i\}_{i\in I}\subseteq\tau_1\cap\tau_2$. We have by definition:

$$\{U_i\}_{i\in I}\subseteq\tau_1 \quad \text{and} \quad \{U_i\}_{i\in I}\subseteq\tau_2, \qquad (\star)$$

which individually yield:

$$\bigcup_{i\in I}U_i\in\tau_1 \quad \text{and} \quad \bigcup_{i\in I}U_i\in\tau_2$$

and

$$\bigcap_{i\in I}U_i\in\tau_1 \quad \text{and} \quad \bigcap_{i\in I}U_i\in\tau_2,$$

where the intersections of course are allowed only over a finite indexing set. By definition then, we get:

$$\bigcup_{i\in I}U_i\in\tau_1\cap\tau_2 \quad \text{and} \quad \bigcap_{i=1}^{N}U_i\in\tau_1\cap\tau_2.$$

Lastly, $\emptyset, X\in\tau_1\cap\tau_2$, since they are in both components. Therefore $\tau_1\cap\tau_2$ is a topology on X. This holds in particular when $X=\mathbb{R}$.

- -

Since $\bigcap_{\alpha\in A}\tau_\alpha=\tau_{\alpha_0}\cap\Big(\bigcap_{\alpha\neq\alpha_0\in A}\tau_\alpha\Big)$, the induction step reduces to the base case, when $|A|=\aleph_0$.

When $|A|>\aleph_0$, we must consider *transfinite induction*. Which is not covered in this text. I assume the same reduction allows the proof to go through for all *ordinals*. □

Note: Why does this proof outline not work for unions? In the line $(\star)$, containment may fail on both sides (when we switch to "or") by careful choice of the family $\{U_i\}_{i\in I}\in\tau_1\cup\tau_2$. This prevents application of the topological closure for the components.

T P2: Regarding Bases of Closed Sets for Topologies

Statement:
A basis of closed sets for a topology τ on a set X is a collection of closed sets $\mathcal{B}$ in X such that any closed set in X is an (potentially arbitrary) intersection of elements of $\mathcal{B}$.

a.) Show that $\mathcal{B}$ is a basis of closed sets for X if and only if $\{X - B \mid B \in \mathcal{B}\}$ is a basis for τ.

b.) Show that $\mathcal{B}$ is a basis of closed sets for τ if and only if whenever $x \in X$ and F is closed in X such that x is not an element of F, there is an element B of $\mathcal{B}$ that contains F but not x.

c.) Let X be a metrizable space and $\mathbb{R}$ be the set of real numbers with the standard topology. Let $\mathcal{B}$ be the collection of all sets $B \subseteq X$ such that there is a continuous function $f_B : X \to \mathbb{R}$ and an element $x \in \mathbb{R}$ such that $B = f_B^{-1}(\{x\})$. Show that $\mathcal{B}$ is a basis of closed sets for the metric topology on X.

- -

Proof:
a.)
$(\Rightarrow)$:
Suppose $\mathcal{B}$ is a basis of closed sets for (X, τ). Then let $O_{open} \subseteq X$. Since O^c is closed, we have existence of a family $\{B_i\}_{i\in I}$ of closed sets in $\mathcal{B}$ for which:

$$O^c = \bigcap_{i\in I} B_i.$$

Applying the complement and using DeMorgan's Law, we get:

$$O = \bigcup_{i\in I} B_i^c.$$

But each B_i^c is open. So we arrive at the result that any open set in X can be written as a union of open sets of the form $B_i^c = X - B$. Thus, $\{X - B \mid B \in \mathcal{B}\}$ gives a basis for the topology τ on X. □

$(\Leftarrow)$:
Conversely, suppose we have a collection of closed sets $\mathcal{B}$ for which $\{X - B \mid B \in \mathcal{B}\}$ gives a basis for the topology τ on X. Then every open set $O \subseteq X$ can be written as a union of B^c's and applying DeMorgan again, we see that each and every O^c is an intersection of elements of $\mathcal{B}$. Because every closed set is the complement of some open set in X, the result is proven. That is, $\mathcal{B}$ is a basis of closed sets for X. ■

(Next page)

Proof (Continued):

b.) WTS:
[$\mathcal{B}$ is a closed basis for (X, τ)] $\Leftrightarrow$ [$\forall F_{closed} \subseteq X, \forall x \in F^c, \exists B \in \mathcal{B}$, s.t. $F \subseteq B$ and $x \notin B$].

($\Rightarrow$) :
Take a closed basis $\mathcal{B}$, an arbitrary closed $F \subseteq X$ and arbitrary $x \in F^c$. By part (a), we have $\{B^c \mid B \in \mathcal{B}\}$ is a basis for τ so:

$$F^c = \bigcup_{i \in I} B_i^c,$$

for some subcollection of B_i^c's since F^c is open. Now, $x \in F^c$ implies by the definition of union that $\exists j \in I$ for which $x \in B_j^c$. Applying the compliment and DeMorgan's Law:

$$F = \bigcap_{i \in I} B_i,$$

which says $F \subseteq B_i$ for all $i \in I$, particularly $i := j$. Hence $\exists B = B_j \supseteq F$ for which $x \notin B$. $\square$

($\Leftarrow$) :
Suppose the RHS holds for a collection of closed sets $\mathcal{B}$. WTS $\mathcal{B}$ is a closed basis. Accordingly, consider an arbitrary $F_{closed} \subseteq X$. Then we have $\forall x \in F^c$ there exists $B_x \in \mathcal{B}$ containing F but not x.

Since each individual B_x contains F, we have:

$$F \subseteq \bigcap_{x \in F^c} B_x.$$

Now, suppose $y \in F^c$, then by hypothesis there exists $B_y \in \mathcal{B}$ containing F but not y. If we further assume $y \in \bigcap_{x \in F^c} B_x$ as above, then $y \in B_x$ for every $x \in F^c$, in particular: $y \in B_y$, which is a contradiction. Thus, if we hold $y \in \bigcap_{x \in F^c} B_x$ to be true, $y \in F^c$ must be false. So $y \in F$. Arbitrariness of y gives the reverse containment:

$$F \supseteq \bigcap_{x \in F^c} B_x$$

and we're done since we showed an arbitrary closed set in X is an intersection of elements of $\mathcal{B}$, making $\mathcal{B}$ a closed basis. ■

(Next page)

Proof (Continued (2)):

c.) First, observe the following:

∘ Lemma: The function $d(F, x) := \inf_{y \in F}\{d(y, x)\}$ is continuous in x.

Proof: Assume $(x_n \to x) \in X$.

We have by the definition of infimum as a lower bound on $\{d(y, x) \mid y \in F\}$ and the Triangle Inequality, for arbitrary $y \in F$:

$$d(F, x) \leq d(y, x) \leq d(y, x_n) + d(x_n, x).$$

Rearranging this gives:

$$-d(y, x_n) \leq d(x_n, x) - d(F, x).$$

Negating:

$$d(y, x_n) \geq d(F, x) - d(x_n, x).$$

But this true for all $y \in F$ says the RHS of the last line is lower bound for $\{d(y, x_n) \mid y \in F\}$. By definition of infimum as a greatest lower bound, it must be the case that:

$$d(F, x_n) = \inf_{y \in F}\{d(y, x_n)\} \geq RHS = d(F, x) - d(x_n, x).$$

We conclude:

$$d(F, x_n) - d(F, x) \geq -d(x_n, x) \qquad (\bigstar)$$

Similarly, starting instead with $d(F, x_n) \leq d(y, x_n) \leq d(y, x) + d(x, x_n)$, after rearranging and applying Symmetry of the metric, we get:

$$-d(y, x) \leq d(x_n, x) - d(F, x_n).$$

This true for all $y \in F$ says negating and applying infimum definition as greatest lower bound on $\{d(y, x) \mid y \in F\}$ yields:

$$d(F, x) \geq d(F, x_n) - d(x_n, x).$$

Rearranging one last time, we get:

$$d(F, x_n) - d(F, x) \leq d(x_n, x) \qquad (\bigstar\bigstar).$$

Combining $(\bigstar)$ and $(\bigstar\bigstar)$ says:

$$\left|d(F, x_n) - d(F, x)\right| \leq d(x_n, x).$$

Since $x_n \to x$, $d(x_n, x) \to 0$ and hence $d(F, x_n) \to d(F, x)$. We have shown that the function $f(x) := d(F, x)$ preserves limits of sequences and is thus continuous by equivalence of definition.

□

(Continues)

Proof (Continued (3)):

Restating the problem for (c): Let (X, d) and $(\mathbb{R}, d_{std})$ be metric topological spaces. Then define $\mathcal{B}$ as the collection of pullbacks of singletons by continuous functions. WTS $\mathcal{B}$ is a basis of closed sets for the metric topology on X. Let us temporarily write $\beta \in \mathcal{B}$ instead to not confuse them with metric balls.

First of all, $adh(\{x\}) = \{x\}$ since if $B_r(y) \cap \{x\} \neq \emptyset$, then $B_r(y) \cap \{x\} = \{x\}$. Arbitrariness of $r \in \mathbb{R}$ yields $y = x$. So singleton sets in metric spaces are closed. By definition, $f : X \to \mathbb{R}$ is continuous (in the topological sense) if it pullsback open sets to open sets (and likewise closed sets to closed sets), so the $\beta \in \mathcal{B}$ are actually closed. So it makes sense to claim they form a closed basis.

Recall the result of (b):

[$\mathcal{B}$ is a closed basis for (X, τ)] $\Leftrightarrow$ [$\forall F_{closed} \subseteq X$,$\forall x \in F^c$, $\exists B \in \mathcal{B}$, s.t. $F \subseteq B$ and $x \notin B$].

Now, let $F_{closed} \subseteq X$ and take the function:

$$f(x) := d(F, x) = \inf_{y \in F}\{d(y, x)\}.$$

By the Lemma, this function is continuous, has $F \subseteq f^{-1}(\{0\}) =: B$ and $\forall x \in F^c,\ x \notin f^{-1}(\{0\})$. This last fact we know by the compliment being open, we contain x in a nonzero radius metric ball lying entirely in F^c.

We conclude our $\mathcal{B}$ forms a closed basis by going left in the result of (b). ■

Notes: The proof of the Lemma was inspired by [25].

T P3: Regarding Subspaces of Lower-Limit Topology on $\mathbb{R}$

Statement:
Let $\mathbb{R}_l$ be the set of real numbers with the lower-limit topology, that is, the set of half open intervals $[a, b)$ forms a basis for $\mathbb{R}_l$. Suppose $A \subseteq \mathbb{R}$ is a subspace of $\mathbb{R}_l$ such that for every collection $\mathcal{C}$ of closed sets in A, if each finite subset of $\mathcal{C}$ has non-empty intersection then all the sets in $\mathcal{C}$ have a point in common. Prove that A is closed in $\mathbb{R}_l$.

Proof:

T P4: Using Definition of Hausdorff; Working with Cofinite Topology

Statement:
a.) Let X be a topological space. Assume for any $p \neq q$ in X there is a continuous function $f : X \to \mathbb{R}$ so that $f(p) \neq f(q)$. Prove that X is Hausdorff.

b.) Let Y be an infinite set, with the topology so that $U \subseteq Y$ is open if and only if U is empty or $Y - U$ is finite. Prove that every continuous function $f : Y \to \mathbb{R}$ is a constant.

- -

Proof:
a.) Suppose $\forall p \neq q,\ \exists f_{cont.} : X \to \mathbb{R}$ with $f(p) \neq f(q)$. Since $\mathbb{R}$ is metrizable, $f(p) \neq f(q)$ implies for any metric, we have $d(f(p), f(q)) > 0$ [Positive Definite Axiom]. Define for given $p \neq q$, the open metric balls:

$$\widetilde{U} := B_{r/2}(f(p)) \qquad \text{and} \qquad \widetilde{V} := B_{r/2}(f(q)),$$

where $r := d(f(p), f(q))$. Then define:

$$U := f^{-1}(\widetilde{U}) \qquad \text{and} \qquad V := f^{-1}(\widetilde{V}).$$

By continuity of f, U and V are open.

Moreover, we have $p \in U$ (as $f(p)$ is the center of $\widetilde{U}$). Likewise $q \in V$.

Lastly, $U \cap V = \{x \mid f(x) \in \widetilde{U} \cap \widetilde{V}\} = \emptyset$ as $\widetilde{U} \cap \widetilde{V} = \emptyset$.

Arbitrariness of p and q yields X is Hausdorff. $\square$

(Next page)

Proof (Continued):

b.) WTS if $\boldsymbol{Y}$ is an infinite set with the *cofinite topology*, then $\boldsymbol{f : Y \to \mathbb{R}}$ continuous implies $\boldsymbol{f}$ is constant.

Towards contradiction now:

Let $\boldsymbol{f : Y \to \mathbb{R}}$ be continuous and suppose $\boldsymbol{f}$ is *not* constant. Then $\exists \boldsymbol{y_1, y_2} \in \boldsymbol{Y}$ such that $\boldsymbol{f(y_1) \neq f(y_2)}$. By the proof of part (a), we may consider pullbacks of the open balls in $\mathbb{R}$ that separate them (previously called $\boldsymbol{U}$ and $\boldsymbol{V}$).

By continuity, $\boldsymbol{U, V}$ are open. In the *cofinite topology*, this says:

$$\Big[(\boldsymbol{U} = \emptyset) \vee (\boldsymbol{U^c} \text{ is finite})\Big] \wedge \Big[(\boldsymbol{V} = \emptyset) \vee (\boldsymbol{V^c} \text{ is finite})\Big].$$

Since $\boldsymbol{U, V}$ are nonempty, by logical reduction $(\boldsymbol{F} \vee \varphi \vDash \varphi)$ in both conjunction components, we obtain the statement:

$$(\boldsymbol{U^c} \text{ is finite }) \wedge (\boldsymbol{V^c} \text{ is finite}).$$

But as we argued in (a), $\boldsymbol{U \cap V = \emptyset}$, so in particular, $(\boldsymbol{x \in U}) \implies (\boldsymbol{x \notin V})$, arbitrariness of $\boldsymbol{x}$ gives $\boldsymbol{U \subseteq V^c}$. We extract the contradiction from the last statement, since $\boldsymbol{U}$ is infinite and contained in $\boldsymbol{V^c}$, $\boldsymbol{V^c}$ must be infinite. ■

T P5: Connectedness and Continuous Functions

Statement:
a.) Let $\boldsymbol{X}$ be a topological space. Prove that if $\boldsymbol{X}$ is connected, then for every continuous function $\boldsymbol{f : X \to \mathbb{R}}$, its range $\boldsymbol{f(X)}$ is a point or an interval (open, half open-half closed, closed, with end points possibly infinity).

b.) Let $\boldsymbol{Y}$ be a topological space. Assume for every continuous function $\boldsymbol{f : Y \to \mathbb{R}}$, its range $\boldsymbol{f(Y)}$ is a point or an interval. Prove that $\boldsymbol{Y}$ is connected.

Proof: **Restatement of the Problem:**

(a)∧(b): Given $\boldsymbol{(X, \tau)}$ prove:

$$\left(\boldsymbol{X_{connected}}\right) \leftrightarrow \left(\forall \boldsymbol{f_{cont.} : X \to \mathbb{R}}, \quad \boldsymbol{Im(f)} \text{ is an interval}\right).$$

Since the only connected subsets of $\boldsymbol{R}$ are intervals, we state the problem in more generality (provisionally) as:

$$\left(\boldsymbol{X_{connected}}\right) \leftrightarrow \left(\forall \text{continuous } \boldsymbol{f : (X, \tau) \to (Y, \tau')}, \quad \boldsymbol{Im(f)} \text{ is } \textit{connected.}\right). \qquad (\bigstar)$$

We also use the following definition of connectedness (there are other formulations):

- -

• Def A top. space $\boldsymbol{(X, \tau)}$ is **connected** if there *does not exist* a *properly contained subset*, $\boldsymbol{S}$, which is both *open and closed* w.r.t. $\boldsymbol{\tau}$. That is if:

$$\neg\big(\exists \boldsymbol{S \subseteq X : (\emptyset \neq S \neq X)} \wedge (\boldsymbol{S} \text{ is open}) \wedge (\boldsymbol{S} \text{ is closed})\big).$$

- -

(Next Page)

Proof (Continued):

a.) a.k.a. ($\Rightarrow$):
With the above considerations, the proof is by *contrapositive.*

If $\exists\ \boldsymbol{f_{cont.}}$ such that $\boldsymbol{Im(f)}$ is disconnected, then:

$$\exists \boldsymbol{S} \subseteq \boldsymbol{Im(f)}, \text{ such that } \boldsymbol{\emptyset} \neq \boldsymbol{S} \neq \boldsymbol{Im(f)}, \text{ and } \boldsymbol{S} \text{ is open and closed in } \boldsymbol{Y}.$$

Now, by the (by the topological definition of continuity), the pre-image of $\boldsymbol{S}$ is respectively open and closed in $\boldsymbol{X}$.

Moreover, $\left(\boldsymbol{f^{-1}(\emptyset) := \emptyset}\right) \wedge (\boldsymbol{\emptyset \neq S}) \implies \left(\boldsymbol{\emptyset \neq f^{-1}(S)}\right)$.
And $\boldsymbol{(Im(f) := f(X))} \wedge \boldsymbol{(S \neq Im(f))} \implies \left(\boldsymbol{f^{-1}(S) \neq f^{-1}(f(X)) = X}\right)$.

Using $\boldsymbol{f^{-1}(S) \subseteq X}$, we conclude $\boldsymbol{X}$ must be disconnected. $\square$

---

b.) a.k.a. ($\Leftarrow$):

WTS $[\forall(\boldsymbol{f_{cont.} : X \to Y}),\ \boldsymbol{Im(f)_{connected}}] \implies (\boldsymbol{X_{connected}})$.

SKIP [$\times$]

T P6: Distance from Points to Sets in Metric Spaces

Statement:
Let X be a metric space with distance d, and let $A \subset X$ be a nonempty subset. Define the function

$$f(x) = inf\{d(x, a) \mid a \in A\}.$$

a.) Prove that for any $x, y \in X$, $|f(x) - f(y)| \leq d(x, y)$.

b.) Prove that f is continuous.

- -

Proof: Already solved (a) and (b). See the Lemma in Problem T P2(c). □

Notes: This apparently was a freebie for those who knew it was a thing. I had to come up with a function in 2c and it was the only thing that seemed to work. I tried characteristic functions first (they weren't continuous) to make them continuous around arbitrary closed sets the distance function for any particular point didn't quite do it, although in the process, found out that $\mathbf{d(x, y)}$ *was continuous in both variables. Problem being for the pullbacks to work correctly, we needed* $\mathbf{d(F, x)}$. *Here is not explicitly written, but they want* $\mathbf{d(x, A)}$.

T P7: Regarding Denseness and Separability

Statement:
Let $\{0,1\}$ be the set with two elements with discrete topology. Let

$$X = \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \dots$$

Here there are (countably) infinitely many $\{0,1\}$'s and X is equipped with the product topology. Prove that X is separable by finding a countable dense subset in X. (Attention: X is not countable).

- -

Proof:

T P8: On Second Countability

Statement:
Let $\mathbb{R}$ be the set of real numbers with the standard topology, and let $f : \mathbb{R} \to Y$ be a surjective map such that if $B \in Y$ is open in Y then $f^{-1}(B)$ is open in $\mathbb{R}$, and if $A \in \mathbb{R}$ is open in $\mathbb{R}$, then $f(A)$ is open in Y. Show Y is second countable.

- -

Proof:
A generalization of the problem:

Let (X, τ_X) and (Y, τ_Y) be topological spaces and suppose $f : X \to Y$ is a bi-continuous, surjective map. WTS if $\mathcal{B}$ is a basis for τ_X then:

$$f(\mathcal{B}) := \{f(\beta) \mid \beta \in \mathcal{B}\}$$

forms a basis for τ_Y. So we may conclude if X is second countable then so is Y.

Finally to answer the original problem, it would suffice to show that $(\mathbb{R}, \tau_{std.})$ is second countable.

- -

Accordingly, let $U_{open} \subseteq Y = Im(f)$ (surjectivity used here), then $f^{-1}(U)$ is open in X by continuity of f. By definition of basis, $\mathcal{B}$, for τ_X, open sets in X can be written as a union of elements in $\mathcal{B}$. So, we have:

$$f^{-1}(U) = \bigcup_{i \in I} \beta_i,$$

where $\beta_i \in \mathcal{B}$ for some indexing set I.

Mapping forward:

$$U = f(f^{-1}(U)) = f\left(\bigcup_{i \in I} \beta_i\right) = \bigcup_{i \in I} f(\beta_i).$$

The last equality can be shown by opening up the set definitions. Namely: $y \in f\left(\bigcup_{i \in I} \beta_i\right)$ implies $\exists x \in \bigcup_{i \in I} \beta_i$ such that $f(x) = y$. We further have then by definition of the union that $\exists \beta_i$ for which $x \in \beta_i$, this gives of course $y = f(x) \in f(\beta_i) \subseteq \bigcup_{i \in I} f(\beta_i)$, hence $y \in \bigcup_{i n I} f(\beta_i)$. Arbitrariness of y gives the "($\subseteq$)" direction.

For the "($\supseteq$)" direction. Let $y \in \bigcup_{i \in I} f(\beta_i)$. Then $\exists \beta_i$ for which $y \in f(\beta_i)$. Subsequently, $\exists x \in \beta_i \subseteq \bigcup_{i \in I} \beta_i$ for which $y = f(x) \in f(\bigcup_{i \in I} \beta_i)$. $\square$

Now, because of continuity of the pullback map by hypothesis:
$\forall i \in I$, $\beta_{i,open}$ in X implies $f(\beta_i) = (f^{-1})^{-1}(\beta_i)$ is open in Y. By arbitrariness of U, we have shown $f(\mathcal{B})$ is a basis of open sets for τ_Y.

Proof (Continued):

Remark: If f was not surjective, then there might be an open set in Y which can't be covered by such a union of images of basis elements. So the image wouldn't be a basis for τ_Y.

At this point, we may conclude if X is second countable, then so is Y. Since:

$$\Big[|\mathcal{B}| = \aleph_0\Big] \implies \Big[|f(\mathcal{B})| = \aleph_0\Big].$$

$\square$

- -

To complete the proof, we show $(\mathbb{R}, \tau_{std.})$ is second countable.

Claim: $\mathcal{B} := \{B_r(x) \mid r \in \mathbb{Q}^{>0}, x \in \mathbb{Q}\}$ forms a basis for $\tau_{std.}$.

Pf: For any metric space, (X, d), $U_{open} = int(U)$ implies $U = \bigcup\limits_{x \in U} B_{r_x}(x)$, for respective threshold radii r_x keeping each ball contained in U. So the set

$$\mathcal{B}' := \{B_r(x) \mid r \in \mathbb{R}^{>0}, x \in \mathbb{X}\}$$

forms a basis for τ_X.

In the special case of the standard metric on $\mathbb{R}$, we know each ball is just an interval:

$$B_r(x) := (x - r, x + r).$$

Using a lemma that there exists a rational between any two reals (we'll prove below), we have there exists $p \in \mathbb{Q} \cap (x - r, x)$ and $\exists q \in \mathbb{Q} \cap (x, x + r)$. This gives an interval $(p, q) \subseteq (x - r, x + r)$ containing x, with rational center $\frac{p+q}{2}$ and radius $q - \frac{p+q}{2}$. Hence U can be written as a union of these. We conclude that the restriction of $\mathcal{B}$ to metric balls of rational radii and centers forms a basis for the standard topology on $\mathbb{R}$.

Moreover, $|\mathcal{B}| = |\mathbb{Q}^{>0} \times \mathbb{Q}| = \aleph_0$ (result from set theory says countable Cartesian products of countable sets are countable), so $\mathbb{R}$ is second countable as desired. $\square$

- -

Now for the lemma. Treating $\mathbb{R}$ as *Dedekind cuts*, $(\alpha, \beta) \neq \emptyset$ implies $\exists \gamma \in (\alpha, \beta)$. In the order structure correspondence, $\alpha < \gamma < \beta$ implies $\alpha \subset \gamma \subset \beta$ as subsets of $\mathbb{Q}$. The fact that the containment is strict implies $\exists s \in \gamma \backslash \alpha$ and $\exists t \in \beta \backslash \gamma$ (both s and t are rational). $\blacksquare$

T P9: Definition of Normal

Statement:
Let $\mathbb{R}_l$ be the set of real numbers with the lower-limit topology, that is, the set of half open intervals $[a, b)$ forms a basis for $\mathbb{R}_l$. Let A be a subspace of $\mathbb{R}_l$. Prove or disprove that A is necessarily normal.

- -

Proof:

T P10: Familiarity with Quotient Spaces

Statement:
For each of the following equivalence relations on topological spaces, give the familiar space to which the quotient space $X/\sim$ is homeomorphic. You DO NOT need to justify your answer.

a.) Define an equivalence relation on $X = \mathbb{R}^2$ with the standard product topology by setting

$$(x_1, y_1) \sim (x_2, y_2) \quad \text{iff} \quad y_1 - (x_1)^3 = y_2 - (x_2)^3.$$

b.) Define an equivalence relation on $X = \mathbb{R}^2$ with the standard product topology by setting

$$(x_1, y_1) \sim (x_2, y_2) \quad \text{iff} \quad (x_1)^2 + (y_1)^2 = (x_2)^2 + (y_2)^2.$$

c.) Let X be the topological space $\mathbb{R}^2$ with the dictionary order topology. Define an equivalence relation on X by setting

$$(x_1, y_1) \sim (x_2, y_2) \quad \text{iff} \quad x_1 = x_2.$$

d.) Define an equivalence relation on $X = \mathbb{R}$ with the standard topology by setting

$$x \sim y \quad \text{iff} \quad x - y \in \mathbb{Q}.$$

- -

Proof:

a.) This relation draws on the difference coordinates have from being on the cubic curve $y = x^3$. Notice all points on this curve have $y - x^3 = 0$. And so the equivalence class of $(0, 0)$ is the graph of the cubic. Points not satisfying this have say $y - x^3 = c$ for some $c \in \mathbb{R}$. Thus $X\backslash\sim$ is the collection of all vertically translated cubics and should be in 1-1 correspondence with $\mathbb{R}$.

---

b.) This one is also in 1-1 correspondence with $\mathbb{R}$ since this space is the collection of all circles centered about the origin.

---

c.) The dictionary order says two coordinates compare $(x_1, y_1) < (x_2, y_2))$ if either their first coordinates are such that $x_1 < x_2$ or $x_1 = x_2$ and $y_1 < y_2$. Otherwise they are the same coordinates. So the relation here says two elements are in the same equivalence class if their first coordinates agree but not the second. So the quotient space is a bunch of vertical lines parameterized by horizontal distance from zero. So we get $\mathbb{R}$ again.

---

d.) The equivalence class of zero is $\mathbb{Q}$ and any irrational gives an equivalence class different from zero (though not necessarily distinct). Some subset of $\mathbb{R} - \mathbb{Q}$.

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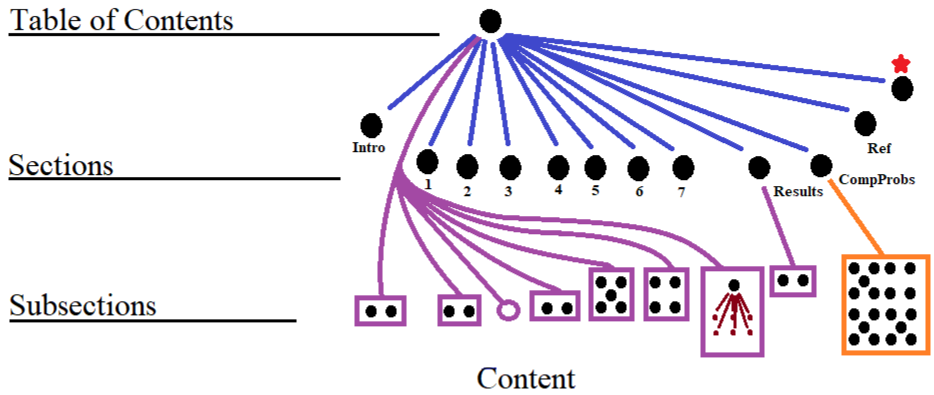

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